Fachbereich Mathematik & Informatik
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Exercise 1 for the lecture

Numerik III

SS 2012

Due: Thursday, April 26th, 2012, 12:15, during the tutorial

Problem 1 (2 TP)
Let $\Omega \subset \mathrm{R}^3$ be a bounded domain with sufficiently smooth boundary and

$$H_C = \{v \in C^1(\overline{\Omega}) \mid v|_{\partial\Omega} = 0\}.$$

Prove that the variational equality

$$u \in H_C: \qquad \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega fv \; dx \qquad \forall v \in H_C$$

has at most one solution.

Problem 2 (4 TP)
Let H be a pre-Hilbert space equipped with the scalar product $(\cdot,\cdot)$ and let ℓ be a bounded, linear functional on H.

a) Show that the quadratic functional

$$\mathcal{J}(v) = \tfrac{1}{2}(v,v) - \ell(v), \quad v \in H,$$

has the Fréchèt derivative

$$\mathcal{J}'(w) = (w,\cdot) - \ell$$

at $w \in H$.

b) Show that $u \in H$ solves the minimization problem

$$u \in H: \qquad \mathcal{J}(u) \leq \mathcal{J}(v) \qquad v \in H,$$

if and only if $\mathcal{J}'(u) = 0$.

c) Apply the abstract results in a) and b) to the special case $H = C^1(\overline{\Omega})$, equipped with the scalar product

$$(v, w)_1 = \int_\Omega \nabla v \cdot \nabla w \; dx + \int_\Omega vw \; dx$$

and $\ell(v) = \int_\Omega fv \; dx$ with $f \in C(\overline{\Omega})$.

d) Determine the Euler-Lagrange equation associated with the above minimization problem in this special case.

Problem 3 (4 PP)
For given $\varepsilon,\ \beta \in \mathrm{R}$ and $\varepsilon > 0$, we consider the boundary value problem to find $u \in C[0,1] \cap C^2(0,1)$ such that

$$\varepsilon u'' + \beta u' = 1 \quad \text{in } (0,1), \qquad u(0) = u(1) = 0. \tag{1}$$

a) Find the solution u of (1).

b) Plot the solution $u =$ for $\varepsilon = 1,\ 10^{-2},\ 10^{-4}$ and $\beta = 1,\ -1$. Which problem is obtained from (1) in the limit case $\varepsilon = 0$?

Problem 4 (2 PP)
For given $g \in \mathrm{R}$, we consider the boundary value problem to find $u \in C[0,1] \cap C^2(0,1)$ such that

$$u'' = 1 \quad \text{in } (0,1), \qquad u'(0) = 0, \;\; u'(1) = g. \tag{2}$$

a) Determine all $g \in \mathrm{R}$ such that (2) has a solution.

b) Let g be selected such that (2) has a solution. Determine a solution that additionally satisfies the condition

$$\int_0^1 u \; dx = M$$

with given $M \in \mathrm{R}$.