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Exercise 10 for the lecture

Numerics III

SoSe 2012

Due: till Thursday, June 28th, 2012, 12 o'clock

Problem 1 (3 TP + 8 PP)
Consider the variational equality

$$u \in H_0^1(\Omega) \qquad a(u,v) = l(v) \qquad \forall v \in H_0^1(\Omega)$$

with a H_0^1-elliptic, symmetric bilinearform a, $l \in (H_0^1(\Omega))'$ and the finite element solutions $u_{\mathcal{S}} \in S^{(1)}$ and $u_{\mathcal{Q}} \in S^{(2)}$. Assume the saturation assumption

$$\exists \beta \in (0,1) \quad : \quad \|u - u_{\mathcal{Q}}\|_a \leq \beta \|u - u_{\mathcal{S}}\|_a \tag{1}$$

is fullfilled.

a) Show the following estimate for the discretezation error:

$$\sqrt{1-\beta^2}\|u - u_{\mathcal{S}}\|_a \leq \|u_{\mathcal{S}} - u_{\mathcal{Q}}\|_a \leq \|u - u_{\mathcal{S}}\|_a.$$

b) Write a Matlab programme `A = assemble_P2(grid, local_assem, Q)`, which assembles the stiffness matrix and the mass matrix for quadratic finite elements, using the corresponding local assemblers and appropriate quadrature rules. Note that there are dgrees of freedom on the edges.

c) Use your programme and the programmes from the homepage to calculate the a posteriori error estimate $\|u_{\mathcal{S}} - u_{\mathcal{Q}}\|_a$ for the solution of the problem given in problem 1d) of exercise 9 on an uniform grid $\mathcal{T}_h$ on $\Omega = (0,1)^2$ with different h. Use the hierarchical P^2 basis here, <u>not</u> the Lagrange basis. Plot the error $\|u_{\mathcal{S}} - u_{\mathcal{Q}}\|_a$ over h in a suitable scale and interprete your results on the background of the a priori error estimates from the lecture.

d) Define the bilinearform

$$b(u, v) = \sum_{E \in \mathcal{E}} u_{x_E} v_{x_E} a(\lambda_E, \lambda_E)$$

in the space $\mathcal{V} = \text{span}\{\lambda_E | E \in \mathcal{E}\}$, where $\mathcal{E}$ is the set of inner edges and λ_E the P^2 basis function associated to midpoint x_E of the edge $E \in \mathcal{E}$. Why can $\tilde{u}_{\mathcal{Q}} = u_{\mathcal{S}} + d$ with

$$d \in \mathcal{V}: \qquad b(d, v) = l(v) - a(u_{\mathcal{S}}, v) \quad \forall v \in \mathcal{V}$$

be interpreted as an inexact evaluation of $u_{\mathcal{Q}}$? Use your programme calculate and plot the error estimate $\|u_{\mathcal{S}} - \tilde{u}_{\mathcal{Q}}\|_a$ analogue to part c). Compare and interprete your results.

Problem 2 (4 TP)
Prove the inverse estimate

$$|v|_1 \leq h^{-1} \|v\|_0$$

for piecewise linear finite element functions $v \in S_h$. Does the estimate also hold for higher order finite element functions?

Problem 3 (4 TP)
Let $\Omega = (a, b)$ and $\mathcal{T}_k$, $k = 0, \ldots, j$, be a sequence of grids as resulting from the successive bisection of the initial grid $\mathcal{T}_0 = \{[a, b]\}$ with mesh size $h_k = (b - a)2^{-k}$. Consider the space $\mathcal{S}_k$ of piecewise affine functions on the intervalls $t \in \mathcal{T}_k$ vanishing at a and b with the nodal basis $\{\lambda_p^{(k)}, p \in \mathcal{N}_k\}$, $k = 0, \ldots, j$. The set of nodes $\mathcal{T}_k$ consists of the set of interior end points of the intervalls $t \in \mathcal{T}_k$. Then the the so-called hierarchical basis Λ^{HB} is defined by

$$\Lambda^{\text{HB}} = \bigcup_{k=1}^{j} \Lambda^{(k)}, \qquad \Lambda^{(k)} = \{\lambda_p^{(k)} \mid p \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}\}.$$

a) Show that the hierarchical basis Λ^{HB} is orthogonal w.r.t. the energy scalar product

$$a(v, w) = \int_{[a,b]} v' w' \, dx$$

on $H_0^1(\Omega)$.

b) Use the results from a) to derive an exact solver for the linear system associated with the finite element approximation

$$u_h \in \mathcal{S}_h: \qquad a(u_h, v) = l(v) \qquad \forall v \in \mathcal{S}_h.$$

c) Is there a relation to the method of cyclic reduction?