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Exercise 11 for the lecture

Numerics III

SoSe 2012

Due: till Thursday, July 5th, 2012, 12 o'clock

Problem 1 (2 TP)
Consider the smoothing property

$$\langle A_k v, v\rangle \le \omega_0 \langle B_k v, v\rangle \qquad \forall v \in \mathrm{R}^{n_k} \tag{1}$$

for symmetric positive definite matrices $A_k, B_k \in \mathrm{R}^{n_k}$.

a) Show that the smoothing property implies

$$\lambda_{\max}(B_k^{-1} A_k) \le \omega_0.$$

b) Show that the sequence u_k^ν generated by

$$B_k(u_k^{\nu+1} - u_k^\nu) = b_k - A_k u_k^\nu$$

converges to the solution u_k of $A_k u_k = b_k$ if (1) holds with $\omega_0 < 2$.

Problem 2 (4 TP)
Show that the preconditioner of the symmetric Gauß–Seidel method given by

$$B_k = (D_k + L_k) D_k^{-1} (D_k + R_k)$$

satisfies the smoothing property (1) for the symmetric positive definite matrix $A = D + L + R$ with $\omega_0 = 1$.

Problem 3 (4 TP + 8 extra PP)

a) Show that the multilevel Gauß-Seidel method is equivalent to the multigrid V-cycle (algorithm 6.2 in the lecture notes).

b) Derive an algebraic representation of the multigrid V-cycle.

c) Implement a multigrid V-cycle for the Poisson problem

$$\begin{aligned} -\Delta u &= 1 \qquad \text{on } [0,1], \\ u(0) = u(1) &= 0. \end{aligned} \tag{2}$$

As a smoother implement a Gauß–Seidel as well as a Jacobi method.

d) Calculate the exact solution u^* of (2). Use a V-cycle with Gauß–Seidel smoother and a V-cycle with Jacobi smoother, respectively to calculate an approximative solution for the initial iterate $u_0 = 0$ on a grid with 6 levels and 64 elements. Plot the error $e_i = \|u_i - u^*\|_A$ as a function of the iteration step i. Based on these results estimate the convergence rates and plot them as a fuction of the number of grid levels for grids with up to 12 levels. Are the convergence rates bounded from above by a constant $c < 1$ or do they converge to 1 for finer grids?