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Exercise 3 for the lecture

Numerics III

SoSe 2012

Due: till Thursday, May 10th, 2012, 12 o'clock

Problem 1 (6 TP)
Let us consider the heat equation in one space dimension.

a) Find an ill-posed initial value problem for the backward facing heat equation

$$u_t + u_{xx} = 0$$

in $\mathbb{R} \times \mathbb{R}^+$, for which the solution does not continuously depend on the data.

b) Let $\sum_{j=0}^{\infty} a_j \cos(j\pi x)$ be the Fourier series of the 2-periodic function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$ on $[-1, 1]$. Derive the (classical) solution (there is only one) of the initial value problem for the heat equation on $\mathbb{R} \times \mathbb{R}^+$ with an initial distribution f in terms of series. What can you say about the quality (smoothness) of the solution $u(\cdot, t)$ for arbitrary $t > 0$ compared to f? What does that mean for the backward facing heat equation? Interprete this fact physically, especially concerning the behavior of $u(\cdot, t)$ for $t \to \infty$.

Problem 2 (4 TP)
Let $\Gamma := \{e^{i\phi} \mid 0 < \phi < \alpha\}$ and $\Omega := \{rz \mid 0 < r < 1, z \in \Gamma\}$. Solve the Poisson equation

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \\ u(e^{i\phi}) &= \sin\left(\phi \frac{\pi}{\alpha}\right) && \text{on } \Gamma \\ u &= 0 && \text{on } \partial\Omega \backslash \Gamma \, . \end{aligned}$$

For which $\alpha \in]0, 2\pi]$ do we have $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, and for which $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$?
Hint: Try the ansatz $u = z^a$.

Problem 3 (6 TP)

a) Compute the singularity function $s(\cdot, a)$ in one dimension, in order to be able to proof the representation formula for the Poisson problem in this case, analogously to the lecture notes. Notice that for a function f on the interval $[b, c]$, one defines

$$\int_{\partial[b,c]} f(x) d\sigma(x) := f(b) + f(c) \ ,$$

and for the outer normals, we have $n(b) = -1$ and $n(c) = 1$.

b) Derive for the interval $[0, L]$, with $L > 0$, Green's function of the first kind $G(\cdot, a)$, which yields a representation formula for solutions $u \in C^2([0, L])$ of the Dirichlet problem

$$-u_{xx} = f \text{ on } [0, L] \ , \quad u(0) = g(0) \ , \quad u(L) = g(L) \ ,$$

in which, additionally to G, the data f and g have to be considered.