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Exercise 4 for the lecture

Numerics III

SoSe 2012

Due: till Wednesday, May 16th, 2012, 12 o'clock

Problem 1 (2 TP)
Show that the Laplace operator is invariant under rotations of the coordinate system, i.e. given an orthogonal matrix A with columns $a_i \in \mathrm{R}^d$, show that

$$\Delta u(x) = (\Delta(u \circ \psi))(\psi^{-1}x) = \Delta_A u(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial a_i \partial a_i} u(x) \quad \forall x \in \mathrm{R}^d$$

where ψ is a rotation given by $\psi(x) = A(x - c) + c$ around a center $c \in \mathrm{R}^d$.

Problem 2 (4 TP)
Consider a domain $\Omega \subset \mathrm{R}^d$ and $G : (\overline{\Omega} \times \Omega) \setminus \{(x,a) | x = a\} \to \mathrm{R}$ such that each $G(\cdot, a)$ is a Green's function of first kind. Show that G is strictly positiv, i.e.,

$$G(x,a) > 0 \qquad \forall x, a \in \Omega, x \neq a$$

by using a Maximum principle.

Problem 3 (4 PP)
Illustrate the influence of pertubations of the right hand side f in x_0 on the solution $u(x)$ of the poisson problem

a) on the interval $\Omega_1 = [0, L] \in \mathrm{R}$

b) on the disc $\Omega_2 = K_R(y) \in \mathrm{R}^2$, here the Green's function is given by

$$G(x,a) = -\frac{1}{2\pi}\left(\log|x-a| - \log|\left(\frac{|a-y|}{R}|x-a'|\right)\right)$$

with $x, a \in \Omega_2$ and $a' = y + R^2|a-y|^{-2}(a-y)$.

Problem 4 (4 extra TP)
The difference star of the nine-point formula is given by

$$\Delta_9 u(x) = \sum_{i,j=-1}^{1} S_{i+2,j+2}\, u\left(x + \begin{pmatrix} i \\ j \end{pmatrix} h\right) \qquad \text{with} \qquad S = \frac{1}{6h^2}\begin{pmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{pmatrix}.$$

Show that under sufficient regularity assumptions, for $\Delta u = \text{const}$ it holds that

$$\Delta u(x) - \Delta_9 u(x) = \mathcal{O}(h^4).$$