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Exercise 5 for the lecture

Numerics III

SoSe 2012

Due: till Thursday, May 24th, 2012, 12 o'clock

Problem 1 (2 TP)
Find a domain Ω and a grid Ω_h with constant mesh size h, such that Ω is connected but Ω_h is not discrete connected.

Problem 2 (3 TP)
Show that the weights $\alpha_Z, \alpha_W, \alpha_O, \alpha_N, \alpha_S$ of the standard five-point finite difference approximation of the Laplace operator satisfy

$$\alpha_Z < 0, \quad \alpha_W, \alpha_O, \alpha_N, \alpha_S > 0, \quad \alpha_Z + \alpha_W + \alpha_O + \alpha_N + \alpha_S = 0.$$

Problem 3 (8 PP)
Approximate the solution of the following BVP conditions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega_i \\ u &= 0 \quad \text{on } \partial\Omega_i \ , \end{aligned}$$

$i = 1, 2$, using the Shortley-Weller method (finite differences):

a) Let $\Omega_1 = (0,1) \times (0,1)$ and $f = 1$ on the disc around the point $(0.5, 0.5)$ with radius $r = 0.3$, and $f = 0$ elsewhere. Calculate numerically the approximate solution for different mesh sizes and compare your results with a reference solution (on a very fine grid, e.g. with step size $h = 1/50$). Which order of accuracy does your method have in the L^2 norm and in the L^∞ norm?

b) Consider the Dirichlet problem from above on the domain $\Omega_2 = \Omega_1 \backslash \Omega$, where $\Omega = (0.5, 1) \times (0.5, 1)$. What do you observe?

Problem 4 (5 TP + 3 extra PP)
Consider the boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g_D && \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= g_N && \text{on } \Gamma_N \end{aligned}$$

on the domain $\Omega = (0,1) \times (0,1)$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$.

a) Derive a finite difference approximation for the case
$\Gamma_D = \{(x_1, x_2) | 0 < x_1 \le 1, x_2 = 1\} \cup \{(x_1, x_2) | x_1 = 1, 0 < x_2 \le 1\}$ using
(i) the standard five-point formula
(ii) the nine-point formula (exercise 4, problem 4).

b) Implement your discretisation from part ii) and solve the problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \frac{\partial u(x_1, 0)}{\partial n} &= -4\pi \sin(4\pi x_1) && x_1 \in (0,1) \\ \frac{\partial u(x_2, 0)}{\partial n} &= -4\pi \sin(4\pi x_2) && x_2 \in (0,1) \end{aligned}$$

with the exact solution $u(x_1, x_2) = \sin(4\pi x_1) \sin(4\pi x_2)$.
Plot the point errors at $(h, h), \ldots, (1-h, 1-h)$ for different mesh sizes h and discuss the results.