Department of Mathematics & Computer Science
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Maren-Wanda Wolf

Exercise 8 for the lecture

Numerics III

SoSe 2012

Due: till Thursday, June 14th, 2012, 12 o'clock

Problem 1 (4 TP)
Let $\Omega \subset \mathbb{R}^n$ be a domain with a sufficiently smooth boundary. Consider the boundary value problem

$$\begin{aligned} -\alpha\Delta u + \beta \cdot \nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $\alpha > 0$, $\beta \in \mathbb{R}^n$ and a given function $f \in C(\overline{\Omega})$.

a) Derive a weak formulation of this problem and show, that the resulting variational problem is well posed in the Sobolev space $H_0^1(\Omega)$.

b) How does the variational problem change, if β is a function in $C^1(\overline{\Omega})^n$, and under which condition do we have still a well posed problem?

Problem 2 (3 TP)
Let $\Omega \subset \mathrm{R}^n$ be a boundet domain and $\partial_D \cup \partial_N = \partial\Omega$ a non-overlapping partition of $\partial\Omega$. Consider the function

$$u = u_0 + w_0,$$

where $w_0 \in H^1(\Omega)$ satisfies $\mathrm{tr}_{\partial_D} w_0 = g_D$ and $u_0 \in H = \{v \in H^1(\Omega) | \mathrm{tr}_{\partial_D} v = 0\}$ solves

$$a(u_0, v) = l(v) - a(w_0, v) \quad \forall v \in H$$

with

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \quad \text{and} \quad l(v) = \int_\Omega fv + \int_{\partial_N} g_N v \quad g_N \in L^2(\partial_N).$$

Show that u solves

$$\begin{aligned} -\Delta u(x) &= f(x) && \forall x \in \Omega, \\ u(x) &= g_D && \forall x \in \partial_D, \\ \frac{\partial}{\partial n} u(x) &= g_N && \forall x \in \partial_N \end{aligned}$$

if $u \in C^2(\overline{\Omega})$.

Problem 3 (2 TP)
Consider the grid $a = x_0 < x_1 < \ldots < x_n = b$ on the interval (a, b). Let u be the solution of

$$u \in H_0^1(a,b): \qquad (u', v') = (f, v) \qquad \forall v \in H_0^1(a,b)$$

and $u_h \in S_h$ the approximation in the linear finite element space $S_h \subset H_0^1(a,b)$ on the above grid. Show that u_h coincides with the linear interpolation of u on the grid, i.e. $u_h(x_i) = u(x_i)$ for $i = 0, \ldots, n$.

Problem 4 (6 PP)
To solve a variational problem in a finite dimensional space V we need to assemble a matrix $A \in \mathrm{R}^{n \times n}$ with $A_{i,j} = a(\lambda_i, \lambda_j)$ where $\{\lambda_1, \ldots, \lambda_n\}$ is a basis of V.

a) Make yourself familiar with the MATLAB programms `basis.m` and `quadrature.m` on the homepage.

b) Write a MATLAB programm `A = assemble_stiff(S, B, Q)`, which assembles the bilinearform

$$\int_\tau \nabla u(x) \cdot \nabla v(x) dx$$

for the basis $\{\lambda_{i,\tau}\}$ on a triangle τ given by the columns of the matrix `S`. The basis $\{\lambda_{i,\tau}\}$ on τ results from a transformation of the basis given by `B` on the unit simplex. Use the quadrature rule `Q` for evalution of the integrals.
Test your programm with `B = basis(1)`, `Q = quadrature(1)` and three triangles of your choice. What happens, if you scale one triangle with different factors h?

c) Analog to b) write a MATLAB programm `M = assemble_mass(S, B, Q)`, which assembles the bilinearform

$$\int_\tau u(x) v(x) dx.$$

Add a Gauß quadratur rule to `quadrature.m`, which is of order $p \geq 2$ on the unit simplex in R^2. Test your programm with `B = basis(1)` and the new quadratur rule `Q = quadrature(2)` and triangles of your choice. What happens, if you scale one triangle with different factors h?