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Exercise 9 for the lecture

Numerics III

SoSe 2012

Due: till Thursday, June 21th, 2012, 12 o'clock

Problem 1 (8 PP)

a) Make yourself familiar with the Matlab programmes `basis.m`, `quadrature.m` and `uniform_grid.m` on the homepage.

b) Write a Matlab programme `A = assemble_P1(grid, local_assem, Q)`, which assembles the global matrix $A_{i,j} = a(\lambda_j, \lambda_i)$ for the linear finite elements nodal basis $\{\lambda_i\}$ on the grid `grid`. The matrix should be calculated as a sum of element matrices assembled by the function `M = local_assem(T, B, Q)`. Thereby the columns of `T` give a triangle of the grid, `B = basis(1)` the local basis and `Q` a quadrature rule defined on the unit simplex. Test your programme by assembling the stiffness and the mass matrix for a uniform grid, using the local assemblers `assemble_stiff` and `assemble_mass` and appropriate quadrature rules.

c) Write a Matlab programme `[A,b] = assemble_dirichlet(grid, A, b, g)`, which „includes"Dirichlet boundary conditions, given by the function `function y = g(x)`, in the matrix `A` and the right-hand side `b`.

d) Use your programmes to approximate a solution of the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

for

$$f(x) = \begin{cases} 0.2 & \text{for } |x - (0.5, 0.5)| \leq 0.2 \\ 0 & \text{else} \end{cases}$$

and $g = 0$ with linear finite elements on the unit square $\Omega = [0,1]^2$ and on the unit circle $\Omega = K_1(0)$, and visualize the solution with the Matlab command `trisurf`.

Advices:

- You can load a grid on the unit circle with the command `grid = load('circle')`, using the file 'circle' on the homepage.
- The right-hand side b can be assembled by linear interpolation of f, i.e. by evaluation at the grid points and multiplication with the mass matrix.

Problem 2 (8 TP)

a) Consider $F_\tau = B_\tau + p_0 : T \to \tau$ with B_τ linear, the transformation of the unit triangle $T = \operatorname{conv}\{0, e_1, e_2\}$ on a triangle $\tau = \operatorname{conv}\{p_0, p_1, p_2\} \subset \mathrm{R}^2$. Show the estimates

$$|v|_{1,\tau} \le c_2 |B_\tau^{-1}|\, |\det B_\tau|^{\frac{1}{2}}\, |v \circ F_\tau|_{1,T},$$
$$|v \circ F_\tau|_{2,T} \le c_1 |B_\tau|^2\, |\det B_\tau|^{-\frac{1}{2}}\, |v|_{2,\tau}$$

by using the chain rule for $v \in H^2(\tau)$.

b) Consider $F = B_\tau + p_0$ as defined in a). Show that

$$|B_\tau| \le (2 + \sqrt{2}) r_\tau, \qquad\qquad |B_\tau^{-1}| \le \frac{1}{\sqrt{2}} \rho_\tau^{-1}$$

with r_τ and ρ_τ the radii of the outer and inner circle of τ.

c) Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a family of triangulations of a domain $\Omega \subset \mathrm{R}^2$ with polygonal boundary. Prove the following error estimate for the interpolation operator $I_h : C(\Omega) \to S_h$

$$\|u - I_h u\|_{1,\Omega} \le \left(c \max_{\tau \in \mathcal{T}_h} \frac{r_\tau}{\rho_\tau} \right) h |u|_2 \qquad \forall u \in C^2(\Omega).$$

You can use the estimate

$$\|v - I_T v\|_{1,T} \le c |v|_{2,T}$$

for functions v and the interpolation I_T on the unit triangle.