Ergodic Theory and Transfer Operators — Question sheet 1 —

Summer 2015

Exercise 1 (Sequence of sets)
Show that if $\{A_n\}$ is an increasing sequence of measurable sets with $A_n \uparrow A$, then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

Hint: Consider the disjoint differences $A_n \setminus A_{n-1}$.

Exercise 2 (Circle rotation: invariant measure)
Let $T : S^1 \to S^1$ denote the rigid rotation of the circle of unit circumference by angle α; that is the map $x \mapsto x + \alpha$. Let m denote the Lebesgue measure on S^1, and $\mathcal{B}$ the Borel σ-algebra .
Show that for every $A \in \mathcal{B}$ one has $m(T^{-1}A) = m(A)$. Proceed as follows:

(a) Show the claim for intervals $I \subset S^1$.

(b) Conclude from (a) that the claim holds for finite unions of intervals $E = \bigcup_{i=1}^n I_i$, I_i interval, $n \in \mathbb{N}$.

(c) Conclude that the claim holds for the algebra $\mathcal{A}$ of subsets of S^1 generated by all intervals.

(d) Consider $\mathcal{M} = \{E \in \mathcal{B} \,|\, m(T^{-1}E) = m(E)\}$. Show that $\mathcal{M}$ is a monotone class.

(e) Use the Monotone Class Theorem (cf. Handout 1) to show the claim for any $A \in \mathcal{B}$.

Exercise 3 (Invariant sets)
Let $T : X \to X$ some transformation, and let $E \in \mathcal{B}$ be an invariant set, i.e. $T^{-1}(E) = E$. Prove that its complement, $E^c = X \setminus E$, is also an invariant set.

Exercise 4 (MATLAB: ergodicity vs mixing)
Consider the two maps on $X = S^1 \times S^1$:

$$T_1(x,y) = \begin{pmatrix} x + \sqrt{2} \\ y + \sqrt{3} \end{pmatrix} \mod 1, \qquad T_2(x,y) = \begin{pmatrix} x + y \\ x + 2y \end{pmatrix} \mod 1\,.$$

(a) Choose a random point $x \in X$, and plot the points $\{T_1^k x\}_{k=0}^{10^4}$ and then $\{T_2^k x\}_{k=0}^{10^4}$. Do the trajectories seem to "get everywhere" for both systems? Do you see a difference in the density of points for the both systems?

(b) Now, choose 1000 random points in $[0, 0.1] \times [0, 0.1]$, we denote this set of points by E. Plot $T_i^1(E), T_i^2(E), \ldots, T_i^7(E)$ for $i = 1, 2$. What kind of qualitative difference do you observe?

Remark: The observed difference is the difference between "ergodic" and "mixing" behavior, and is going to be characterized in the lectures.

Exercise 5 (MATLAB: do you trust the results?)
Let $T : S^1 \to S^1$ denote the circle doubling map: $x \mapsto 2x \mod 1$. Write a MATLAB program to simulate 60 iterates of T, i.e. the set $\{T^k x\}_{k=0}^{60}$ for some initial $x \in S^1$. Is the answer as you expected?