Ergodic Theory and Transfer Operators
— Question sheet 2 —

Summer 2015

Exercise 1 (Delta measure)
Let $(X, \mathcal{B})$ be a measurable space and define the *delta measure* at $x \in X$ as

$$\delta_x(A) := \begin{cases} 1, & x \in A; \\ 0, & \text{otherwise.} \end{cases} \qquad \text{for all } A \in \mathcal{B}.$$

(a) Show that δ_x is a probability measure.

(b) If x is a *fixed point* of a measurable map $T : X \to X$, i.e. $Tx = x$, show that δ_x is an invariant probability measure for T.

(c) Generalizing (b), if x is a *periodic point of period* $p \in \mathbb{N}$, i.e. $T^p x = x$, show that

$$\mu = \frac{\delta_x + \delta_{Tx} + \ldots + \delta_{T^{p-1}x}}{p}$$

is an invariant probability measure for T.

Exercise 2 (Measurability)
Let $(X, \mathcal{B}, \mu)$ be a measure space, $T : X \to X$ a measurable transformation, and let $E \in \mathcal{B}$ be given. Consider the set $E_0 \subset X$ where E_0 is the set of all states which visit E infinitely often:

$$E_0 := \left\{ x \in X \,\middle|\, T^k(x) \in E \text{ for infinitely many } k \geq 0 \right\}.$$

Show that E_0 is measurable and invariant.

Hint: To show measurability, first show that for $n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{N}$ the set $\left\{ x \in X \,\middle|\, T^k(x) \in E \text{ precisely for } k = k_1, k_2, \ldots, k_n \right\}$ is measurable, then consider the complement of E_0.

Exercise 3 (Ergodicity and essential reachability)
Let $(X, \mathcal{B}, \mu, T)$ be a ppt and $A \in \mathcal{B}$ with $\mu(A) > 0$. Consider $D_A := \bigcup_{n=0} T^{-n}A$. Show that if T is ergodic, then $\mu(D_A) = 1$; that is the set of points which reach A under iteration of T is of full measure.

Hint: What can you say about $D_A \triangle T^{-1}D_A$?

Exercise 4 (Mixing: an invariant under measure-theoretic isomorphism)
Show that (strong) mixing is an invariant of measure theoretic isomorphism. That is, suppose $(X, \mathcal{B}, \mu, T)$ is a mixing ppt, and that $(X', \mathcal{B}', \mu', T')$ is a ppt which is measure theoretically isomorphic to $(X, \mathcal{B}, \mu, T)$. Show that $(X', \mathcal{B}', \mu', T')$ is mixing.

Exercise 5 (MATLAB: circle rotation)
Let $T = R_\alpha : S^1 \to S^1$, $x \mapsto x + \alpha \mod 1$, denote the rigid rotation of the unit circle by angle α. Choose $\alpha = 1/\pi$.

(a) Write a program to simulate 60 iterates of T.

(b) Plot $T^k x$ vs k.

(c) Plot $T^k x$ vs $T^{k-1}x$.

(d) Select a subinterval on the circle and test Poincaré's Recurrence Theorem by verifying that orbits continue returning into that subinterval for as long as you care to simulate.

(e) Plot $\{T^k x\}_{k=0}^{10^5}$ as points in $[0,1)$ for various initial conditions x. Based on the results, can you decide whether or not T is ergodic?

(f) Define $f, g : S^1 \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = \sin(\pi x/2)$. Numerically estimate

$$\int f \cdot g \circ T^k \, dm - \int f \, dm \cdot \int g \, dm$$

for $k = 1, \ldots, 100$, where m is the Lebesgue measure on S^1. Based on the results, can you decide whether T is mixing?

Exercise 6 (Recurrence for images)
Consider the sequence of images seen here.[1] It is generated by applying the same permutation over and over to the pixels of an image. Iterating this procedure, after some (possibly many) steps the original image occurs. Explain this behavior by using Poincaré's Recurrence Theorem.

[1] http://en.wikipedia.org/wiki/Arnold%27s_cat_map#/media/File:Arnold%27s_Cat_Map_animation_(74px,_zoomed,_labelled).gif