Ergodic Theory and Transfer Operators
— Question sheet 3 —

Summer 2015

Exercise 1 (Bernoulli measure)
Let $X = \{0,1\}^{\mathbb{N}}$, $\mathcal{B}$ the corresponding σ-algebra generated by the cylinders, and μ the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure. Show that the set

$$X' = \big\{\{x_k\} \in X \,\big|\, \nexists n \text{ s.t. } x_\ell = 1 \; \forall \ell \geq n\big\}$$

is μ-measurable and has full measure.

Hint: Show that the complement of X' is a countable set.

Remark: This result is used in the lectures (§1.18) to show the measure-theoretic isomorphy of the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli scheme and the angle doubling map.

Exercise 2 (Subshift of finite type: topological properties)
Prove that the subshift of finite type with the metric $d(\{x_k\}, \{y_k\}) = 2^{-\min\{k \,|\, x_k \neq y_k\}}$ is a compact metric space, and that the shift operator on it is continuous.

Exercise 3 (Characterization of mpts)
Let $(X, \mathcal{B}, \mu)$ be a measure space and $T : X \to X$ a measurable transformation. Show that the following are equivalent:

(a) $(X, \mathcal{B}, \mu, T)$ is a mpt.

(b) For any $f \in L^\infty(X, \mu)$ it holds
$$\int_X f d\mu = \int_X f \circ T d\mu\,.$$

Hint: Simple functions are dense in L^∞.

Exercise 4 (The tent map)
Represent the tent map $T : [0,1] \to [0,1]$, $T(x) = 1 - 2|x - \frac{1}{2}|$, on $\{0,1\}^{\mathbb{N}}$.

Exercise 5 (Markov measure)
Let P be a stochastic matrix and p a stationary, componentwise positive, probability vector. Let μ denote the (p, P)-Markov measure, T the left shift, and $p_{ij}^{(n)}$ the $(i,j)^{\text{th}}$ entry of P^n. Further, let $[\underline{a}] = [a_0, \ldots, a_k]$ and $[\underline{b}] = [b_0, \ldots, b_\ell]$. Show that for $n > k$

$$\mu\left([\underline{a}] \cap T^{-n}[\underline{b}]\right) = p_{a_0} p_{a_0 a_1} \cdots p_{a_{k-1} a_k} p_{a_k b_0}^{(n-k)} p_{b_0 b_1} \cdots p_{b_{\ell-1} b_\ell}\,.$$