ERGODIC THEORY AND TRANSFER OPERATORS
— QUESTION SHEET 4 —

Summer 2015

Exercise 1 (Product)
Verify Proposition 1.21 (ergodicity and mixing for products of mpts) for an example numerically. Using the circle rotation $T_1(x) = x + \alpha \mod 1$, and the angle tripling map $T_2(x) = 3x \mod 1$, look at $T_1 \times T_1$, $T_2 \times T_2$, and $T_1 \times T_2$.
What do you expect: is the product of an ergodic transformation and a mixing transformation always ergodic/mixing?

Exercise 2 (Skew-product)
Let $(\Omega, \mathcal{F}, \varrho, \sigma)$ be a ppt and $\{(X, \mathcal{B}, \mu, T_\omega)\}_{\omega\in\Omega}$ be a family of ppts. Define the skew-product $\tau(\omega, x) = (\sigma\omega, T_\omega x)$ as in the lectures.

(a) For $F \in \mathcal{F}$, $A \in \mathcal{B}$, show τ preserves the product measure $\varrho \times \mu$ of sets of the form $F \times A$; that is, show that $(\varrho \times \mu)(\tau^{-1}(F \times A)) = (\varrho \times \mu)(F \times A)$.

Remark: This is the first step in showing that τ preserves the product measure $\varrho \times \mu$ of all measurable sets in $F \otimes \mathcal{B}$.

(b) Show that the skew-product is an extension of its base $(\Omega, \mathcal{F}, \varrho, \sigma)$; that is, the base is a factor of the skew-product.

Exercise 3 (Induced transformation)
Describe the action of the induced transformation T_A when

(a) $X = [0, 1)$, $T(x) = x + \alpha \mod 1$, and $A = [0, 1/2)$;

(b) $X = \{0, 1\}^{\mathbb{N}}$, σ is the left shift and $A = \{x \in X \mid x_0 = 0\}$.

Exercise 4 (Markov chain)
Suppose one has 10 coins, numbered from 1 to 10, all showing heads. Every minute we uniformly randomly choose a number between 1 and 10, and turn over the coin with that number. This is a Markov process on 11 states, where the current state is the number of tails (thus we start in state 0).

(a) Describe the 11×11 stochastic transition matrix containing the conditional transition probabilities between these states.

(b) Determine the stationary distribution on the 11 states, i.e. the left-invariant probability vector to the stochastic matrix from (a).

(c) Using Kac's lemma, determine the expected time to return to an "all heads" configuration, if one starts in an "all heads" configuration.

Hint: translate the problem to one about Markov shifts.