Ergodic Theory and Transfer Operators
— Question sheet 5 —

Summer 2015

Exercise 1 (Validating the Birkhoff Ergodic Theorem)
Choose an initial point $x_0 \in S^1$, and compute an orbit of length 10^6 using the circle tripling map T.

(a) Calculate how many points in the orbit fall in the interval $[0, 1/10)$. Is this consistent with the Birkhoff Ergodic Theorem?

(b) Use the Matlab command `hist` to plot a histogram of the orbit of length 10^6 on 1000 bins. Is this consistent with the Birkhoff Ergodic Theorem?

(c) Calculate $f_n := \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x_0)$, $n = 1, \ldots, 10^6$, for $f(x) = x^2$. Plot f_n vs. n. Is this consistent with the Birkhoff Ergodic Theorem?

Exercise 2 (Ergodicity and Extremality)
An invariant probability measure μ is called *extremal*, if it cannot be written in the form $\mu = t\mu_1 + (1-t)\mu_2$, where μ_1 and μ_2 are two different invariant probability measures, and $0 < t < 1$. Prove that an invariant probability measure is extremal if and only if it is ergodic, using the following steps.

(a) Show that if $E \in \mathcal{B}$ is a T-invariant set of nonzero measure, then the measure μ_E, defined by $\mu_E(A) := \mu(E \cap A)/\mu(E)$ for every $A \in \mathcal{B}$, is T-invariant. Deduce that if μ is not ergodic, then it is not extremal.

(b) Show that if μ is ergodic, and $\mu = t\mu_1 + (1-t)\mu_2$, where μ_1 and μ_2 are invariant, and $0 < t < 1$, then

   (i) For every $E \in \mathcal{B}$, it holds $\frac{1}{n}\sum_{k=0}^{n-1} \chi_E \circ T^k(x) \to \mu(E)$ as $n \to \infty$ for μ_i-a.e. $x \in X$ ($i = 1, 2$).

   Hint: Show that $\mu(A) = 0$ implies $\mu_i(A) = 0$ ($i = 1, 2$) for $A \in \mathcal{B}$.

   (ii) Conclude that $\mu_i(E) = \mu(E)$ for all $E \in \mathcal{B}$ ($i = 1, 2$).

   Hint: Dominated convergence theorem.

Exercise 3 (Strong Law of Large Numbers)
Prove the strong law of large numbers for the following experiment:
Throw a fair dice successively (throws are independent), and let X_i denote the value of the i^{th} throw. Show that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 3.5 \quad \text{as } n \to \infty, \text{ almost surely,}$$

by using Birkhoff's Ergodic Theorem for a suitable Bernoulli shift.