Ergodic Theory and Transfer Operators
— Question sheet 7 —

Summer 2015

Exercise 1 (Conditional expectation)
Let $(X, \mathcal{B}, \mu)$ be a measure space and $\mathcal{A} \subset \mathcal{B}$ another σ-algebra. Show that

(a) $f \mapsto \mathbb{E}(f|\mathcal{A})$ is linear and a contraction in the L^1-metric;

(b) if $f \geq 0$, then $\mathbb{E}(f|\mathcal{A}) \geq 0$;

(c) if $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then $\mathbb{E}(\phi \circ f|\mathcal{A}) \leq \phi\big(\mathbb{E}(f|\mathcal{A})\big)$;

(d) if g is $\mathcal{A}$-measurable, then $\mathbb{E}(gf|\mathcal{A}) = g\,\mathbb{E}(f|\mathcal{A})$;

(e) if $\mathcal{F} \subset \mathcal{A}$ is a σ-algebra, then $\mathbb{E}\big(\mathbb{E}(f|\mathcal{A})|\mathcal{F}\big) = \mathbb{E}(f|\mathcal{F})$;

(f) $\text{esssup} f \geq \text{esssup}\mathbb{E}(f|\mathcal{A})$; and

Hint: For an $\mathcal{A}$-measurable function f holds $\text{esssup} f > M$ for some $M \in \mathbb{R} \Leftrightarrow$ there exists an $A \in \mathcal{A}$ with $\mu(A) > 0$ such that $f > M$ a.e. on A.

(g) if $X = [-1, 1]$, $\mathcal{B}$ the Borel σ-algebra, μ the Lebesgue measure, and $\mathcal{A} = \{A \in \mathcal{B} \mid A = -A\}$, then $\mathbb{E}(f|\mathcal{A})(x) = \frac{1}{2}\big(f(x) + f(-x)\big)$.

Exercise 2 (The FPO acting on densities)
Let $T : [0, 1) \to [0, 1)$ be the circle tripling map $T(x) = 3x \mod 1$, and let P be the associated FPO (with respect to the Lebesgue measure).

(a) Compute Pf and $P^2 f$ for $f = 3\chi_{[0,1/3)}$ and for $f = 2\chi_{[0,1/2)}$.

(b) Verify your computations by representing the respective density f through a large number of uniformly distributed initial states which you then iterate once and twice by T, and plot the distribution of the image points.

Exercise 3 (Fixed point of the FPO)
Let $T : [0, 1) \to [0, 1)$ be given by $T(x) = 4x(1 - x)$, and let P be the associated FPO (w.r.t. Lebesgue). Show that $f^*(x) = \frac{1}{\pi\sqrt{x(1-x)}}$ is a fixed point of P.

Exercise 4 (Representation of the FPO)
Let $X \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a connected open set, and $T : X \to X$ diffeomorphism; that is T is invertible and both T and T^{-1} are differentiable. Show that the FPO (w.r.t. Lebesgue) associated with T satisfies

$$Pf(x) = f\big(T^{-1}(x)\big)\big|\det DT^{-1}(x)\big|,$$

with D denoting the derivative.