Ergodic Theory and Transfer Operators
— Question sheet 8 —

Summer 2015

Exercise 1 (Bounded variation)

Show Theorem 3.8A from the lectures; that is:

(a) If f is of BV on $I = [a, b]$, then f is bounded on I; in fact it holds that

$$|f(x)| \le f(a) + V_I f \quad \forall x \in I\,.$$

(b) If f is of BV, and $\|f\|_{L^1} < \infty$, then

$$|f(x)| \le V_I f + \frac{\|f\|_{L^1}}{b-a} \quad \forall x \in I\,.$$

Exercise 2 (Transfer operators and the support)

Let $T : X \to X$ a non-singular transformation, P and U the associated Frobenius–Perron, and Koopman operators, respectively. Further, let $f \in L^1(X)$ and $g \in L^\infty(X)$. Show that

(a) $\operatorname{supp} Ug = T^{-1}(\operatorname{supp} g)$; and

(b) $\operatorname{supp} Pf \subseteq T(\operatorname{supp} f)$,

where $\operatorname{supp} h$ denotes the support of the function h, i.e. the set $\{h \neq 0\}$ (which is thus defined up to a null set).

Hint: For (b), assume the counterpart, i.e. that $M := \operatorname{supp} Pf \setminus T(\operatorname{supp} f) \neq \emptyset$, and consider Pf on subsets of M.

(c) Give an example where the inclusion in (b) is strict.

Exercise 3 (A useful partition)

Let $T : [0, 1] \to [0, 1]$ be the tent transformation, i.e.

$$T(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 1 - 2x, & \frac{1}{2} \le x \le 1\,. \end{cases}$$

Let P be the Frobenius–Perron operator associated with T. Let $\mathcal{S}$ be the set of all functions f of the form $f = \alpha\chi_{[0,\frac{1}{2}]} + \beta\chi_{(\frac{1}{2},1]}$, where $\alpha, \beta \in \mathbb{R}$.

(a) For $f \in \mathcal{S}$ write $f = (\alpha, \beta)$, whenever $f = \alpha\chi_{[0,\frac{1}{2}]} + \beta\chi_{(\frac{1}{2},1]}$. Show that $Pf = (\alpha, \beta)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(b) Find an absolutely continuous invariant measure of T.

Exercise 4 (Non-finite invariant measure)

(a) Let $T : [a, b] \to [a, b]$ be a piecewise monotonic transformation. Prove Proposition 3.7; that is, that the associated Frobenius–Perron operator w.r.t. the Lebesgue measure satisfies

$$Pf(x) = \sum_{z \in T^{-1}(x)} \frac{f(z)}{|T'(z)|} \quad \text{a.e.}$$

Hint: Generalize the procedure we used in Example 3.3.

(b) Let

$$T(x) = \begin{cases} \frac{x}{1-x}, & 0 \le x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x \le 1\,. \end{cases}$$

Show that $f(x) = \frac{1}{x}$ is a fixed point of P.