Ergodic Theory and Transfer Operators
— Question sheet 9 —

Summer 2015

Exercise 1 (Nonlinear change of coordinates)
Let $T : I \to I$ be non-singular, and let P_T be the Frobenius–Perron operator of T. Let $h : I \to I$ be a diffeomorphism (that is, h is a bijection and both h and its inverse h^{-1} are differentiable on $(0, 1)$). Let $S : I \to I$ be given by $S(x) = h \circ T \circ h^{-1}(x)$, and let P_S be the Frobenius–Perron operator of S. Show that:

(a) $P_T f = f$ implies $P_S g = g$, where $g = (f \circ h^{-1}) \cdot |(h^{-1})'|$;

For the rest of the problem suppose f is a T-invariant density.

(b) Show that g is an S-invariant density.

(c) Let μ_f be the measure such that $\frac{d\mu_f}{dm} = f$, and define μ_g similarly. Show that the Lyapunov exponent of the ppt $(I, \mathcal{B}, \mu_f, T)$ coincides with the Lyapunov exponent of the ppt $(I, \mathcal{B}, \mu_g, S)$. (The Lyapunov exponent of a differentiable ppt $(X, \mathcal{A}, \mu, T)$ is defined as follows: $\int_X \log |T'(x)| d\mu$.)

Exercise 2 (Change of measure)
Let μ and ν be equivalent probability measures; that is $\mu \ll \nu$ and $\nu \ll \mu$. Show, that if $h \in \mathcal{D}(X, \mathcal{B}, \nu)$ is the Radon–Nikodým derivative of μ w.r.t. ν, then for any $f \in L^1(X, \mathcal{B}, \mu)$

$$P_{T,\mu} f = \frac{P_{T,\nu}(fh)}{h},$$

where $P_{T,\mu}$ and $P_{T,\nu}$ denote the Frobenius–Perron operator associated with the nonsingular transformation $T : X \to X$ w.r.t. the measures μ and ν, respectively.

Exercise 3 (Variation-decreasing projection)
Let $\mathscr{P} = \{I_1, \ldots, I_n\}$ be a partition of the interval $I \subset \mathbb{R}$, and $\pi_n : L^1 \to V_n$ the linear projection from the lectures from L^1 to $V_n = \text{span}\{\chi_1, \ldots, \chi_n\}$. Show that

$$V_I \pi_n f \leq V_I f .$$

Give an example where $V_I f = \infty$ but $V_I \pi_n f < \infty$.