Ergodic Theory and Transfer Operators
— Question sheet 11 —

Summer 2015

Exercise 1 (Projection revisited)

Let a measurable partition $\mathscr{P}_n = \{B_1, \ldots, B_n\}$ of the Lebesgue-measurable space X be given. Show that the projection $\pi_n : L^1 \to \operatorname{span}\{\chi_1, \ldots, \chi_n\}$,

$$\pi_n f = \sum_{i=1}^{n} c_i \chi_i, \quad c_i = \frac{1}{m(B_i)} \int_{B_i} f \, dm,$$

is a non-expansion, that is $\|\pi_n f\|_{L^1} \leq \|f\|_{L^1}$ for every $f \in L^1$.

Exercise 2 (Entropy of partitions)

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let ξ and η be finite partitions of X. Let $H(\xi)$ denote the entropy of the partition ξ. Show:

(a) $H(\xi) \geq 0$, and equality holds only if and only if $\xi = \{X\}$.

(b) If $\xi \leq \eta$, then $H(\xi) \leq H(\eta)$, and equality holds if and only if $\xi = \eta$.

(c) If ξ has n elements, then $H(\xi) \leq \log n$, and equality holds if and only if the measure of each element of ξ is $1/n$.

Exercise 3 (Conditional entropy)

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let ξ, η, and ζ be finite partitions of X, such that $\eta \leq \zeta$. Show that $H(\xi|\eta) \geq H(\xi|\zeta)$.