ERGODIC THEORY AND TRANSFER OPERATORS
— HANDOUT 2 —
Summer 2015

# On the convergence of Ulam's method

In §3.15 of the lectures we discussed Li's result [Li76]:

**Theorem 3.15.** Let $T : I \to I$ be a piecewise expanding transformation with $\inf |T'| > 1$. Suppose the associated Frobenius–Perron operator $P$ has a unique invariant density $h \in \mathcal{D}(I, \mathcal{B}, m)$. Let

$$\mathscr{P}_n = \{B_1^{(n)}, \ldots, B_n^{(n)}\}, \quad n = 1, 2, \ldots$$

be partitions of $I$ such that $\max_{1\le i\le n} m(B_i^{(n)}) \to 0$ as $n \to \infty$, and $k \ge 1$ sufficiently large. Further, let $f_n \in \mathcal{D}(I, \mathcal{B}, m)$ be a fixed density of $P_n^k := \pi_n \circ P^k$. Then

$$f_n^* := \frac{1}{k}\sum_{i=0}^{k-1} P^i f_n \to h \quad \text{as } n \to \infty \text{ in } L^1.$$

Somewhat unsatisfactory about the statement of Theorem 3.15 is that we have no access to $P$ and its powers, only to discrete approximations $P_n^i$ for (theoretically) arbitrary $i \in \mathbb{N}$. Here we show a remedy to this fact, in particular

**Corollary 3.15.** In the setting of Theorem 3.15, we have that

$$\frac{1}{k}\sum_{i=0}^{k-1} (\pi_n P)^i f_n \to h \quad \text{as } n \to \infty \text{ in } L^1.$$

The idea of the proof is going to be to show that difference between $\frac{1}{k}\sum_{i=0}^{k-1}(\pi_n P)^i f_n$ and $\frac{1}{k}\sum_{i=0}^{k-1} P^i f_n$ vanishes as $n \to \infty$. To this end, the following lemma will turn out to be useful. It can be easily shown by induction over $k$.

**Lemma 1:** For any $f \in L^1$,

$$\sum_{i=0}^{k-1}(\pi_n P)^i f - \sum_{i=0}^{k-1} P^i f = \sum_{i=1}^{k-1}\sum_{j=0}^{i-1}(\pi_n P)^j(\mathrm{id} - \pi_n)P^{i-j} f, \tag{1}$$

where id denotes the identity operator.

*Proof of Corollary 3.15.* Consider the right hand side of equation (1) with $f = f_n$. Our strategy is to show that each term in this finite sum converges to zero as $n \to \infty$. Let us recall the following facts from the lectures:

(a) $\|f_n\|_{L^1} = 1$, since they are densities, and $V_I f_n \le K$ for some $K \le \frac{\beta}{1-\alpha}$, as shown in §3.15. Here, $\alpha$ and $\beta$ were the constants from the Lasota–Yorke inequality for $P^k$.

(b) The Lasota–Yorke inequality (Lemma 3.9A) also gives us constants $\alpha_i, \beta_i > 0$ such that

$$V_I P^i f \le \alpha_i V_I f + \beta_i \|f\|_{L^1}.$$

(c) By Helly's selection theorem (Theorem 3.8C), any subset $\mathcal{L}$ of $L^1$ bounded in the BV-norm (i.e. there is some $C < \infty$ such that $\|f\|_{BV} \le C$ for every $f \in \mathcal{L}$) is relatively compact[1] in $L^1$.

[1] A set $\mathcal{L} \subset L^1$ is relatively compact if its closure in $L^1$ is compact.

Combining (a) and (b) yields that

$$V_I P^i f_n \leq \bar{\alpha} K + \bar{\beta},$$

where $\bar{\alpha} = \max_{0\leq i \leq k-1} \alpha_i$ and $\bar{\beta} = \max_{0\leq i\leq k-1} \beta_i$. Hence, the sequence $\{P^{i-j} f_n\}_{n\in\mathbb{N}}$ is bounded in variation for every $i = 1, \ldots, k-1$ and $j = 0, \ldots, i-1$, thus relatively compact in $L^1$ by (c), since $\|P^{i-j} f_n\|_{L^1} = \|f_n\|_{L^1} = 1$. By Lemma 2 below we have that $(\mathrm{id} - \pi_n)P^{i-j} f_n \to 0$ as $n \to \infty$ for every $i = 1, \ldots, k-1$ and $j = 0, \ldots, i-1$. It can be easily seen that $\|\pi_n\|_{L^1} \leq 1$. Then, since $\|\pi_n P\|_{L^1} \leq \|P\|_{L^1} = 1$, every term in the sum on the right hand side of (1) goes to zero, and hence the sum itself. This concludes the proof. □

**Lemma 2:** Let $\mathcal{L} \subset L^1$ be relatively compact in $L^1$. Then for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that

$$\|(\mathrm{id} - \pi_n) f\|_{L^1} < \varepsilon$$

for every $n \geq N$ and $f \in \mathcal{L}$.

In fact, it holds that every projection which converges pointwise to the identity on some Banach space, converges uniformly to the identity on relatively compact subsets of that Banach space.

*Proof.* Let $B_\varepsilon(f) := \{g \in L^1 \mid \|f - g\|_{L^1} < \varepsilon\}$ denote the $\varepsilon$-ball in $L^1$ around $f$. Relative compactness of $\mathcal{L}$ means that for any given $\varepsilon > 0$ there is a finite set $\mathcal{L}_\varepsilon = \{f_1, \ldots, f_{k_\varepsilon}\} \subset \mathcal{L}$ such that

$$\mathcal{L} \subset \bigcup_{i=1}^{k_\varepsilon} B_\varepsilon(f_i)\,.$$

This is the finite subcover property of compact sets. Now, since $\pi_n$ converges pointwise, we can find $N_\varepsilon \in \mathbb{N}$ such that $\|(\mathrm{id} - \pi_n) f_i\|_{L^1} < \varepsilon$ for every $i = 1, \ldots, k_\varepsilon$ and $n \geq N_\varepsilon$. Then, for every $f \in \mathcal{L}$ we have that

$$\|f - \pi_n f\|_{L^1} \leq \|f - f_i\|_{L^1} + \|f_i - \pi_n f_i\|_{L^1} + \|\pi_n f_i - \pi_n f\|_{L^1},$$

and for some $f_i \in \mathcal{L}_\varepsilon$ each of the three terms on the right hand side is bounded by $\varepsilon$; the last one due to $\|\pi_n\|_{L^1} \leq 1$. This concludes the proof. □

## Literatur

[Li76] Tien-Yien Li. Finite approximation for the Frobenius–Perron operator. A solution to Ulam's conjecture. *Journal of Approximation theory*, 17(2):177–186, 1976.