

Exercise 10 for the lecture
NUMERICS III
SoSe 2015

Due: till Tuesday, 30. June

Problem 1 (8 programming points)

Consider the variational equality

$$u \in H_0^1(\Omega) \quad a(u, v) = l(v) \quad \forall v \in H_0^1(\Omega)$$

with a H_0^1 -elliptic, symmetric bilinearform a , $l \in (H_0^1(\Omega))'$ and the finite element solutions $u_S \in S^{(1)}$ and $u_Q \in S^{(2)}$. Assume the saturation assumption

$$\exists \beta \in (0, 1) \quad : \quad \|u - u_Q\|_a \leq \beta \|u - u_S\|_a \quad (1)$$

is fulfilled.

a) Show the following estimate for the discretization error:

$$\sqrt{1 - \beta^2} \|u - u_S\|_a \leq \|u_S - u_Q\|_a \leq \|u - u_S\|_a.$$

- b) Write a MATLAB programme `A = assemble_P2(grid, local_assem, Q)`, which assembles the stiffness matrix and the mass matrix for quadratic finite elements, using the corresponding local assemblers and appropriate quadrature rules. Note that there are degrees of freedom on the edges.
- c) Use your programme and the programmes from the homepage to calculate the a posteriori error estimate $\|u_S - u_Q\|_a$ for the solution of the problem given in problem 1d) of exercise 9 on an uniform grid \mathcal{T}_h on $\Omega = (0, 1)^2$ with different h . Use the hierarchical P^2 basis here, not the Lagrange basis. Plot the error $\|u_S - u_Q\|_a$ over h in a suitable scale and interpret your results on the background of the a priori error estimates from the lecture.

d) Define the bilinearform

$$b(u, v) = \sum_{E \in \mathcal{E}} u_{x_E} v_{x_E} a(\lambda_E, \lambda_E)$$

in the space $\mathcal{V} = \text{span}\{\lambda_E | E \in \mathcal{E}\}$, where \mathcal{E} is the set of inner edges and λ_E the P^2 basis function associated to midpoint x_E of the edge $E \in \mathcal{E}$. Why can $\tilde{u}_Q = u_S + d$ with

$$d \in \mathcal{V} : \quad b(d, v) = l(v) - a(u_S, v) \quad \forall v \in \mathcal{V}$$

be interpreted as an inexact evaluation of u_Q ? Use your programme calculate and plot the error estimate $\|u_S - \tilde{u}_Q\|_a$ analogue to part c). Compare and interpret your results.

Problem 2

Prove the inverse estimate

$$|v|_1 \leq ch^{-1} \|v\|_0$$

for piecewise linear finite element functions $v \in S_h$. Does the estimate also hold for higher order finite element functions?

Problem 3

Let $\Omega = (a, b)$ and $\mathcal{T}_k, k = 0, \dots, j$, be a sequence of grids as resulting from the successive bisection of the initial grid $\mathcal{T}_0 = \{[a, b]\}$ with mesh size $h_k = (b - a)2^{-k}$. Consider the space \mathcal{S}_k of piecewise affine functions on the intervals $t \in \mathcal{T}_k$ vanishing at a and b with the nodal basis $\{\lambda_p^{(k)}, p \in \mathcal{N}_k\}, k = 0, \dots, j$. The set of nodes \mathcal{T}_k consists of the set of interior end points of the intervals $t \in \mathcal{T}_k$. Then the the so-called hierarchical basis Λ^{HB} is defined by

$$\Lambda^{\text{HB}} = \bigcup_{k=1}^j \Lambda^{(k)}, \quad \Lambda^{(k)} = \{\lambda_p^{(k)} | p \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}\}.$$

a) Show that the hierarchical basis Λ^{HB} is orthogonal w.r.t. the energy scalar product

$$a(v, w) = \int_{[a,b]} v'w' dx$$

on $H_0^1(\Omega)$.

b) Use the results from a) to derive an exact solver for the linear system associated with the finite element approximation

$$u_h \in \mathcal{S}_h : \quad a(u_h, v) = l(v) \quad \forall v \in \mathcal{S}_h.$$

Problem 4 (extra points)

Consider the smoothing property

$$\langle A_k v, v \rangle \leq \omega_0 \langle B_k v, v \rangle \quad \forall v \in \mathbb{R}^{n_k} \quad (2)$$

for symmetric positive definite matrices $A_k, B_k \in \mathbb{R}^{n_k}$.

- a) Show that the smoothing property implies

$$\lambda_{\max}(B_k^{-1}A_k) \leq \omega_0.$$

- b) Show that the sequence u_k^ν generated by

$$B_k(u_k^{\nu+1} - u_k^\nu) = b_k - A_k u_k^\nu$$

converges to the solution u_k of $A_k u_k = b_k$ if (2) holds with $\omega_0 < 2$.

Problem 5 (extra points)

- a) Show that the multilevel Gauß-Seidel method is equivalent to the multigrid V-cycle (algorithm 6.2 in the lecture notes).
- b) Derive an algebraic representation of the multigrid V-cycle.
- c) Implement a multigrid V-cycle for the Poisson problem

$$\begin{aligned} -\Delta u &= 1 && \text{on } [0, 1], \\ u(0) = u(1) &= 0. \end{aligned} \quad (3)$$

As a smoother implement a Gauß-Seidel as well as a Jacobi method.

- d) Calculate the exact solution u^* of (3). Use a V-cycle with Gauß-Seidel smoother and a V-cycle with Jacobi smoother, respectively to calculate an approximative solution for the initial iterate $u_0 = 0$ on a grid with 6 levels and 64 elements. Plot the error $e_i = \|u_i - u^*\|_A$ as a function of the iteration step i . Based on these results estimate the convergence rates and plot them as a function of the number of grid levels for grids with up to 12 levels. Are the convergence rates bounded from above by a constant $c < 1$ or do they converge to 1 for finer grids?