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Exercise 4 for the lecture

Numerics III

SoSe 2015

Due: till Tuesday, 19. May

Problem 1

a) Let $\sum_{j=0}^{\infty} a_j \cos(j\pi x)$ be the Fourier series of the 2-periodic function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$ on $[-1, 1]$. Derive the (classical) solution (there is only one) of the initial value problem for the heat equation on $\mathbb{R} \times \mathbb{R}^+$ with an initial distribution f in terms of series. What can you say about the quality (smoothness) of the solution $u(\cdot, t)$ for arbitrary $t > 0$ compared to f?

b) What difficulties arise if one considers the backward heat equation instead? How could these be physically interpreted?

Problem 2

We consider the following initial-boundary value problem for the backward heat equation

$$u_t + \pi^{-2} u_{xx} = 0 \quad \text{on } (0,1)\times(0,T), \quad u(0,t) = u(1,t) = 0, \quad u(x,0) = \sin(\pi x) \text{ for } x \in (0,1). \tag{1}$$

a) Find the exact solution of (1). Sow that (1) is ill-posed for $T = \infty$.

b) Derive a discrete analogue of (1) for $T = 10 < \infty$ by replacing the partial derivatives by finite differences:

$$u_t \approx \frac{1}{\Delta t}(u(x_i, t_{k+1}) - u(x_i, t_k)), \qquad u_{xx} \approx \frac{1}{h^2}(u(x_{i+1}, t_k) - 2u(x_i, t_k) + u(x_{i-1}, t_k))$$

with $h = 1/N$, and $\Delta t = T/M$. Compute the resulting numerical approximation $U_{ik} \approx u(x_i, t_k)$, $i = 0, \ldots, N$, $k = 0, \ldots, M$ for various N, $M < 100$ and compare the results with the exact solution.

Problem 3

a) Compute the singularity function $s(\cdot, a)$ in one dimension, in order to be able to proof the representation formula for the Poisson problem in this case, analogously to the lecture notes. Notice that for a function f on the interval $[b, c]$, one defines

$$\int_{\partial[b,c]} f(x) d\sigma(x) := f(b) + f(c) ,$$

and for the outer normals, we have $n(b) = -1$ and $n(c) = 1$.

b) Derive for the interval $[0, L]$, with $L > 0$, Green's function of the first kind $G(\cdot, a)$, which yields a representation formula for solutions $u \in C^2([0, L])$ of the Dirichlet problem

$$-u_{xx} = f \text{ on } [0, L] , \quad u(0) = g(0) , \quad u(L) = g(L) ,$$

in which, additionally to G, the data f and g have to be considered.

Problem 4

Consider a domain $\Omega \subset \mathrm{R}^d$ and $G : (\overline{\Omega} \times \Omega) \setminus \{(x, a) | x = a\} \to \mathrm{R}$ such that each $G(\cdot, a)$ is a Green's function of first kind. Show that G is strictly positiv, i.e.,

$$G(x, a) > 0 \qquad \forall x, a \in \Omega, x \neq a$$

by using a Maximum principle.