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Exercise 5 for the lecture

# Numerics III

SoSe 2015

**Due: till Tuesday, 26. May**

**Problem 1**

Show that the weights $\alpha_Z, \alpha_W, \alpha_O, \alpha_N, \alpha_S$ of the standard five-point finite difference approximation of the Laplace operator satisfy

$$\alpha_Z < 0, \quad \alpha_W, \alpha_O, \alpha_N, \alpha_S > 0, \quad \alpha_Z + \alpha_W + \alpha_O + \alpha_N + \alpha_S = 0.$$

**Problem 2**

The difference star of the nine-point formula is given by

$$\Delta_9 u(x) = \sum_{i,j=-1}^{1} S_{i+2,j+2} u\left( x + \begin{pmatrix} i \\ j \end{pmatrix} h \right) \qquad \text{with} \qquad S = \frac{1}{6h^2} \begin{pmatrix} -1 & -4 & -1 \\ -4 & 20 & -4 \\ -1 & -4 & -1 \end{pmatrix}.$$

Show that under sufficient regularity assumptions, for $\Delta u = \text{const}$ it holds that

$$\Delta u(x) - \Delta_9 u(x) = \mathcal{O}(h^4).$$

**Problem 3** (8 programming points)

Approximate the solution of the following BVP conditions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega_i \\ u &= 0 \quad \text{on } \partial\Omega_i \ , \end{aligned}$$

$i = 1, 2$, using the Shortley-Weller method (finite differences):

a) Let $\Omega_1 = (0,1) \times (0,1)$ and $f = 1$ on the disc around the point $(0.5, 0.5)$ with radius $r = 0.3$, and $f = 0$ elsewhere. Calculate numerically the approximate solution for different mesh sizes and compare your results with a reference solution (on a very fine grid, e.g. with step size $h = 1/50$). Which order of accuracy does your method have in the $L^2$ norm and in the $L^\infty$ norm?

b) Consider the Dirichlet problem from above on the domain $\Omega_2 = \Omega_1 \backslash \Omega$, where $\Omega = (0.5, 1) \times (0.5, 1)$. What do you observe?

**Problem 4**

Consider the boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= g_D && \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= g_N && \text{on } \Gamma_N. \end{aligned}$$

Derive a consistent finite difference approximation for an arbitrary domain $\Omega$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, where the normal $n$ is given for all $x \in \partial\Omega$.