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Exercise 7 for the lecture

Numerics III

SoSe 2015

Due: till Tuesday, 9. June

Problem 1

Which of the following functions is an element of $H^1((-1,1))$ (and why or why not, respectively)?

a) $x \mapsto \sqrt{x+1}$

b) $x \mapsto |x|$

c) $x \mapsto |x| + \chi_{\mathbb{Q}}(x)$

Here, $\chi_{\mathbb{Q}}$ is the characteristic function of the set $\mathbb{Q}$, i.e., it is 1 on $\mathbb{Q}$ and 0 otherwise.

Problem 2

Let $\Omega \subset \mathbb{R}^n$ be a domain with a sufficiently smooth boundary. Consider the boundary value problem

$$\begin{aligned} -\alpha\Delta u + \beta \cdot \nabla u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $\alpha > 0$, $\beta \in \mathbb{R}^n$ and a given function $f \in C(\overline{\Omega})$.

a) Derive a weak formulation of this problem and show, that the resulting variational problem is well posed in the Sobolev space $H_0^1(\Omega)$.

b) How does the variational problem change, if β is a function in $C^1(\overline{\Omega})^n$, and under which condition do we have still a well posed problem?

Problem 3

Let $\Omega \subset \mathrm{R}^n$ be a boundet domain and $\partial_D \cup \partial_N = \partial\Omega$ a non-overlapping partition of $\partial\Omega$. Consider the function

$$u = u_0 + w_0,$$

where $w_0 \in H^1(\Omega)$ satisfies $\mathrm{tr}_{\partial_D} w_0 = g_D$ and $u_0 \in H = \{v \in H^1(\Omega) | \mathrm{tr}_{\partial_D} v = 0\}$ solves

$$a(u_0, v) = l(v) - a(w_0, v) \quad \forall v \in H$$

with

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \quad \text{and} \quad l(v) = \int_\Omega fv + \int_{\partial_N} g_N v \quad g_N \in L^2(\partial_N).$$

Show that u solves

$$\begin{aligned} -\Delta u(x) &= f(x) & \forall x \in \Omega, \\ u(x) &= g_D & \forall x \in \partial_D, \\ \frac{\partial}{\partial n} u(x) &= g_N & \forall x \in \partial_N \end{aligned}$$

if $u \in C^2(\overline{\Omega})$.