HIERARCHICAL ERROR ESTIMATES FOR THE LAPLACIAN

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1. Error estimates from an extended space

We consider the continuous problem

$$u \in H: \qquad a(u,v) = (f,v) \qquad \forall v \in H = H_0^1(\Omega) \tag{1.1}$$

with $a(v,w) = (\nabla v, \nabla w)$, $f \in L^2(\Omega)$, the L^2-scalar product $(\cdot,\cdot)$ and $\Omega \in \mathbb{R}^2$ with polygonal boundary. We introduce the energy norm

$$\|v\| = a(v,v)^{1/2}.$$

Suppose $\mathcal{T}$ is a conforming triangulation of Ω. Then $\mathcal{S}$ denotes the space of continuous functions that are piecewise affine over $\mathcal{T}$ and vanish on $\partial\Omega$. The space $\mathcal{S}$ is spanned by the nodal basis $\{\phi_P \mid P \in \mathcal{N} \cap \Omega\}$, where $\mathcal{N}$ stands for the set of vertices of $T \in \mathcal{T}$, and the continuous piecewise affine functions ϕ_P associated with $P \in \mathcal{N}$ are characterized by $\phi_P(P') = \delta_{P,P'}$ (Kronecker-δ). The resulting finite element approximation of (1.1) is given by

$$u_{\mathcal{S}} \in \mathcal{S}: \qquad a(u_{\mathcal{S}}, v) = (f,v) \quad \forall v \in \mathcal{S}.$$

We are interested in the a posteriori control of the energy norm $\|u - u_{\mathcal{S}}\|$ of the error $u - u_{\mathcal{S}}$. The error $e = u - u_{\mathcal{S}}$ solves the defect equation

$$e \in H: \qquad a(e,v) = \rho_{\mathcal{S}}(v) = (f,v) - a(u_{\mathcal{S}}, v) \qquad \forall v \in H.$$

The defect equation is approximated by a larger finite element space

$$\mathcal{Q} = \mathcal{S} + \mathcal{V}.$$

The solution $e_{\mathcal{Q}}$ of the discretized defect equation

$$e_{\mathcal{Q}} \in \mathcal{Q}: \qquad a(e_{\mathcal{Q}}, v) = \rho_{\mathcal{S}}(v) \qquad v \in \mathcal{Q}$$

satisfies $e_{\mathcal{Q}} = u_{\mathcal{Q}} - u_{\mathcal{S}}$ with

$$u_{\mathcal{Q}} \in \mathcal{Q}: \qquad a(u_{\mathcal{Q}}, v) = (f,v) \qquad v \in \mathcal{Q}.$$

The quantity $\|e_{\mathcal{Q}}\|$ is our first candidate for an a posteriori error estimate, i.e. for a lower and upper bound for $\|u - u_{\mathcal{S}}\|$.

Saturation assumption versus reliability. Utilizing the Galerkin orthogonality

$$a(u - u_{\mathcal{Q}}, v) = 0 \qquad \forall v \in \mathcal{Q}$$

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and $\mathcal{S} \subset \mathcal{Q}$, we obtain

$$\begin{aligned} \|u_{\mathcal{Q}} - u_{\mathcal{S}}\|^2 &= \|u_{\mathcal{Q}} - u + u - u_{\mathcal{S}}\|^2 \\ &= \|u_{\mathcal{Q}} - u\|^2 + \|u - u_{\mathcal{S}}\|^2 + 2a(u_{\mathcal{Q}} - u, u - u_{\mathcal{S}}) \\ &= \|u_{\mathcal{Q}} - u\|^2 + \|u - u_{\mathcal{S}}\|^2 + 2a(u_{\mathcal{Q}} - u, u - u_{\mathcal{Q}}) \\ &= \|u - u_{\mathcal{S}}\|^2 - \|u_{\mathcal{Q}} - u\|^2 \end{aligned} \tag{1.2}$$

This identity immediately implies the lower bound (efficiency)

$$\|u_{\mathcal{Q}} - u_{\mathcal{S}}\| \leq \|u - u_{\mathcal{S}}\|. \tag{1.3}$$

Proposition 1.1. *The saturation assumption*

$$\|u - u_{\mathcal{Q}}\| \leq \beta \|u - u_{\mathcal{S}}\| \qquad \text{with } \beta < 1 \tag{1.4}$$

is equivalent to the upper bound (reliability)

$$\|u - u_{\mathcal{S}}\| \leq C \|u_{\mathcal{Q}} - u_{\mathcal{S}}\| \qquad \text{with } C = 1/(1-\beta)^{1/2}. \tag{1.5}$$

Proof. The assertion is obtained from (1.2) as follows. Assume (1.4). Then

$$\|u - u_{\mathcal{S}}\|^2 = \|u_{\mathcal{Q}} - u_{\mathcal{S}}\|^2 + \|u_{\mathcal{Q}} - u\|^2 \leq \|u_{\mathcal{Q}} - u_{\mathcal{S}}\|^2 + \beta \|u - u_{\mathcal{S}}\|^2$$

implies (1.5).

Assume (1.5). Then

$$\|u_{\mathcal{Q}} - u\|^2 = \|u - u_{\mathcal{S}}\|^2 - \|u_{\mathcal{Q}} - u_{\mathcal{S}}\|^2 \leq (1 - 1/C^2)\|u - u_{\mathcal{S}}\|^2$$

implies (1.4). □

The following principal limitations of (hierarchical) error estimation were first observed by Bornemann et al. [1].

Proposition 1.2. *Assume that the subspace $\mathcal{L} \subset L^2(\Omega)$ satisfies*

$$dim\ \mathcal{L} > dim\ \mathcal{V}.$$

Then there is at least one $f \in \mathcal{L}$, $f \neq 0$, such that $u_{\mathcal{Q}} = u_{\mathcal{S}}$.

Proof. Consider the defect operator

$$D : f \Longrightarrow \mathcal{S}^{\perp}, \quad f \mapsto e_{\mathcal{Q}} = u_{\mathcal{Q}} - u_{\mathcal{S}},$$

where $\mathcal{S}^{\perp}$ denotes the (energy) orthogonal complement of $\mathcal{S}$ in $\mathcal{Q}$. Because of

$$\dim\ \mathcal{L} > \dim\ \mathcal{V} \geq \dim\ \mathcal{S}^{\perp},$$

the operator D cannot be one-to-one, but must have a nontrivial kernel containing the desired $f \neq 0$. □

As a consequence of Proposition 1.2, we can only expect to get reliability up to additional terms, if $f \in L^2(\Omega)$.

2. Quadratic extension, hierarchical splitting, and diagonalization

We select the space $\mathcal{Q} \subset H$ of piecewise quadratic finite elements on $\mathcal{T}$. Let $\mathcal{E}$ denote the set of edges of triangles $T \in \mathcal{T}$. Each function $v \in \mathcal{Q}$ is uniquely determined by its nodal values in $P \in \mathcal{N}_\mathcal{Q} = \mathcal{N} \cup \{x_E \mid E \in \mathcal{E}\}$, where x_E stands for the midpoint of $E \in \mathcal{E}$. Now $\mathcal{Q}$ can be regarded as a hierarchical extension of $\mathcal{S}$, i.e.,

$$\mathcal{Q} = \mathcal{S} + \mathcal{V}, \qquad \mathcal{V} = \text{span}\,\{\phi_E \mid E \in \mathcal{E}\}, \tag{2.6}$$

involving the quadratic bubble functions $\phi_E \in \mathcal{Q}$ characterized by $\phi_E(P) = \delta_{x_E,P}$, $\forall P \in \mathcal{N}_\mathcal{Q}$ (Kronecker-δ).

Remark 2.1. *Our subsequent analysis carries over to hierarchical extensions as spanned by other bubble functions. For example, we could as well define ϕ_E as the piecewise linear nodal basis functions associated with the vertices $x_E \in \mathcal{N}'$ of the triangulation $\mathcal{T}'$ resulting from uniform refinement of $\mathcal{T}$ or, equivalently, select $\mathcal{Q}$ to be the space the piecewise linear finite elements on $\mathcal{T}'$.*

Using the uniquely determined splitting $v = v_\mathcal{S} + v_\mathcal{V}$ and $w = w_\mathcal{S} + w_\mathcal{V}$ of $v, w \in \mathcal{Q}$ into $v_\mathcal{S}, w_\mathcal{S} \in \mathcal{S}$ and $v_\mathcal{V}, w_\mathcal{V} \in \mathcal{V}$, we define the bilinear form

$$a_\mathcal{Q}(v_\mathcal{Q}, w_\mathcal{Q}) = a(v_\mathcal{S}, w_\mathcal{S}) + \sum_{E\in\mathcal{E}} v_\mathcal{V}(x_E) w_\mathcal{V}(x_E) a(\phi_E, \phi_E) \tag{2.7}$$

and the associated energy norm

$$\|v\|_\mathcal{Q} = a_\mathcal{Q}(v,v)^{\frac{1}{2}}$$

on $\mathcal{Q}$. Note that $a_\mathcal{Q}(\cdot,\cdot)$ is resulting from decoupling of $\mathcal{S}$ and $\mathcal{V}$ and subsequent diagonalization on $\mathcal{V}$. The norm equivalence

$$a_\mathcal{Q}(v,v) \sim a(v,v) \qquad \forall v \in \mathcal{Q} \tag{2.8}$$

follows from the estimates

$$\|v_\mathcal{S}\| + \|v_\mathcal{V}\| \sim \|v\|, \quad \|v_\mathcal{V}\|_\mathcal{Q} = \left(\sum_{E\in\mathcal{E}} v_\mathcal{V}(x_E)^2 a(\phi_E, \phi_E) \right)^{\frac{1}{2}} \sim \|v_\mathcal{V}\|, \tag{2.9}$$

as obtained from related local versions [1, 2]

$$\|v_\mathcal{S}\|_T + \|v_\mathcal{V}\|_T \sim \|v\|_T, \quad \left(\sum_{E\in\mathcal{E}_T} v_\mathcal{V}(x_E)^2 a(\phi_E, \phi_E) \right)^{\frac{1}{2}} \sim \|v_\mathcal{V}\|_T, \tag{2.10}$$

where $\mathcal{E}_T$ denotes the set of edges of $T \in \mathcal{T}$.

It has been shown in [2] that the unique solution $e_\mathcal{V}$ of the associated variational equality

$$e_\mathcal{V} \in \mathcal{Q}: \qquad a_\mathcal{Q}(\varepsilon_\mathcal{V}, v) = \rho_\mathcal{S}(v) \qquad \forall v \in \mathcal{Q} \tag{2.11}$$

inherits the norm equivalence (2.8), i.e.,

$$\|e_\mathcal{V}\|_\mathcal{Q} \sim \|e_\mathcal{Q}\|. \tag{2.12}$$

The solution $e_\mathcal{V} \in \mathcal{V}$ is explicitly given by

$$e_\mathcal{V}(p) = 0, \quad p \in \mathcal{N}, \qquad e_\mathcal{V}(x_E) = \frac{\rho_\mathcal{S}(\phi_E)}{\|\phi_E\|^2} \quad E \in \mathcal{E}. \tag{2.13}$$

Finally, the quantity

$$\|\varepsilon_{\mathcal{V}}\|_{\mathcal{Q}} = \eta = \left(\sum_{E\in\mathcal{E}} \eta_E^2\right)^{1/2} \tag{2.14}$$

with

$$\eta_E = |e_{\mathcal{V}}(x_E)|\|\phi_E\| = \frac{\rho_{\mathcal{S}}(\phi_E)}{\|\phi_E\|}, \tag{2.15}$$

is our a posteriori error estimator. The local quantities η are used as local refinement indicators.

The estimator η provides a lower bound for $\|e\|$, as a consequence of (2.12) and (1.3).

On the saturation assumption (1.4), η also provides an upper bound as a consequence of Proposition 1.1 and (2.12). In order to avoid the saturation assumption, we will prove an upper bound directly, up to data oscillation, of course. Using Proposition 1.1, we then show that small data oscillation *implies* the saturation assumption.

3. Global upper bound (reliability)

Green's formula. After integration by parts on each $T \in \mathcal{T}$, the identity $\Delta u_{\mathcal{S}} = 0$ on each T yields the representation

$$\rho_{\mathcal{S}}(v) = \int_\Omega fv + \sum_{E\in\mathcal{E}} \int_E j_E v, \qquad j_E = \partial_{\mathbf{n}} u_{\mathcal{S}}|_{T_2} - \partial_{\mathbf{n}} u_{\mathcal{S}}|_{T_1}. \tag{3.16}$$

Here, $\mathbf{n}$ denotes the unit normal vector on the common edge $E = T_1 \cap T_2$ of two triangles $T_1, T_2 \in \mathcal{T}$ pointing from T_1 to T_2, and $j_E \in \mathbb{R}$ represents the jump of the normal flux associated with $u_{\mathcal{S}}$ across E.

More residuals. In view of the identity

$$\|e\|^2 = \rho_{\mathcal{S}}(e)$$

we will provide an upper bound for $\rho_{\mathcal{S}}(e)$. To this end, we introduce the following localization of $\rho_{\mathcal{S}}$. Invoking the partition of unity

$$\sum_{P\in\mathcal{N}} \phi_P = 1 \qquad \text{in } \Omega, \tag{3.17}$$

we decompose

$$\rho_{\mathcal{S}} = \sum_{P\in\mathcal{N}} \rho_P \tag{3.18}$$

into the local contributions

$$\rho_P(v) = \rho_{\mathcal{S}}(v\phi_P) = \int_{\omega_P} fv\phi_P + \sum_{E\in\mathcal{E}_P} \int_E j_E v\phi_P, \quad v \in H^1(\Omega),$$

where

$$\omega_P = \text{supp } \phi_P, \qquad \mathcal{E}_P = \{E \in \mathcal{E} \mid E \ni P\},$$

denote the support of ϕ_P and the internal edges emanating from P, respectively. Note that we have

$$\rho_P(c) = c\rho_P(1) = c\rho_{\mathcal{S}}(\phi_P) = 0 \qquad \forall P \in \mathcal{N} \cap \Omega. \tag{3.19}$$

for all $c \in \mathbb{R}$.

Local L^2-projections. The reduction of the continuous error $e = u - u_{\mathcal{S}} \in H_0^1(\Omega)$, to its approximation $e_{\mathcal{V}} \in \mathcal{V}$ will be performed by local projections [6]

$$\Pi_P : \ H^1(\Omega) \to \mathcal{Q}_P = \operatorname{span}\{\phi_P\} \cup \mathcal{V}_P, \quad \mathcal{V}_P = \operatorname{span}\ \{\phi_E \mid E \in \mathcal{E}_P\}, \qquad P \in \mathcal{N}.$$

For given $v \in H^1(\Omega)$, the value $\Pi_P v \in \mathcal{Q}_P$ is uniquely defined by the conditions

$$\int_E \Pi_P v = \int_E v \quad \forall E \in \mathcal{E}_P \quad \text{and} \quad \begin{cases} \int_{\omega_P} \Pi_P v = \int_{\omega_P} v & \text{if } P \in \mathcal{N} \cap \Omega, \\ \Pi_P v \in \mathcal{V}_P & P \in \mathcal{N} \cap \partial\Omega. \end{cases} \tag{3.20}$$

It can be verified by straightforward calculations that the coefficients in the hierarchical basis representation

$$\Pi_P v = \alpha_P(v)\phi_P + \sum_{E \in \mathcal{E}_P} \alpha_E(v)\phi_E \tag{3.21}$$

are given by

$$\alpha_P(v) = \begin{cases} \frac{c_P(v)}{c_P(\phi_P)} & \text{if } P \in \mathcal{N} \cap \Omega, \\ 0 & \text{if } P \in \mathcal{N} \cap \partial\Omega, \end{cases} \qquad \alpha_E(v) = \frac{\int_E v - \alpha_P(v) \int_E \phi_P}{\int_E \phi_E}, \tag{3.22}$$

where

$$c_P(v) = \int_{\omega_P} v - \sum_{E \in \mathcal{E}_P} \Big(\int_E v\Big)\Big(\int_{\omega_P} \phi_E\Big)\Big(\int_E \phi_E\Big)^{-1}.$$

In particular, $c_P(\phi_P) = -\frac{1}{6}|\omega_P|$. The following lemma collects some essential properties of the projections Π_P.

Lemma 3.1. *The coefficients in* (3.21) *satisfy*

$$\max_{Q \in \{P\} \cup \mathcal{E}_P} |\alpha_Q(v)| \lesssim h_P^{-1}\big(\|v\|_{0,\omega_P} + h_P \|\nabla v\|_{0,\omega_P}\big) \tag{3.23}$$

and Π_P *is stable in the sense that*

$$\|\Pi_P v\|_{0,\omega_P} \lesssim \|v\|_{0,\omega_P} + h_P \|\nabla v\|_{0,\omega_P}. \tag{3.24}$$

Proof. In order to show (3.23) and (3.24), we start with

$$\Big|\int_{\omega_P} v\Big| \lesssim h_P \|v\|_{0,\omega_P}, \qquad \Big|\int_E v\Big| \leq h_E^{\frac{1}{2}} \|v\|_{0,E} \lesssim h_P (h_P^{-1}\|v\|_{0,\omega_P} + \|\nabla v\|_{0,\omega_P}),$$

where we have used the Cauchy-Schwarz inequality, the 'scaled' trace theorem, and $h_E = |E| \leq h_P$ for $E \in \mathcal{E}_P$. Inserting these estimates and straightforward bounds of the integrals of ϕ_E and ϕ_P in terms of h_P into (3.22), we obtain (3.23). Then (3.24) follows from the triangle inequality, $\|\phi_P\|_{0,\omega_P} \approx h_P$, and $\|\phi_E\|_{0,\omega_P} \approx h_P$. □

Data oscillation. We define

$$\operatorname{osc}(\mathcal{T}, f) = \Bigg(\sum_{P \in \mathcal{N} \cap \Omega} h_P^2 \|f - \bar{f}_P\|_{0,\omega_P}^2 + \sum_{P \in \mathcal{N} \cap \partial\Omega} h_P^2 \|f\|_{0,\omega_P}^2\Bigg)^{1/2}. \tag{3.25}$$

where, for any $P \in \mathcal{N}$, $h_P = \max_{E \in \mathcal{E}_P} |E|$ is a measure for the diameter of ω_P, and

$$\bar{f}_P = \frac{1}{|\omega_p|} \int_{\omega_p} f$$

is the mean value of f on ω_P.

Theorem 3.2. *The upper bound*

$$\|u-u_{\mathcal{S}}\| \le C\Big(\sum_{E\in\mathcal{E}}\eta_E^2+\operatorname{osc}(\mathcal{T},f)^2\Big)^{1/2} \tag{3.26}$$

holds with η_E *defined in* (2.14), $\operatorname{osc}(\mathcal{T},f)$ *defined in* (3.25), *and a constant* C *depending only on the shape regularity of* $\mathcal{T}$.

Proof. We use the identity $\|u-u_{\mathcal{S}}\|^2=\rho_{\mathcal{S}}(e)$ and estimate $\rho_{\mathcal{S}}(e)$. Using the decomposition (3.18) we write $\rho_{\mathcal{S}}(e)=\sum_{P\in\mathcal{N}}\rho_P(e)$. To derive upper bounds for the local contributions $\rho_P(e)$, we distinguish two cases corresponding to the splitting

$$\mathcal{N}=\mathcal{N}\cap\Omega\cup(\mathcal{N}\cap\partial\Omega)$$

which will be addressed in the given order.

Case 1: $P\in\mathcal{N}\cap\Omega$. We claim that

$$\rho_P(e)\lesssim\Big(\sum_{E\in\mathcal{E}_P}\eta_E+h_P\|f-\bar f_P\|_{0,\omega_P}\Big)\|\nabla e\|_{0,\omega_P}. \tag{3.27}$$

In order to prove (3.27), we set

$$w=(e-c)\phi_P,\qquad c=\frac{1}{|\omega_P|}\int_{\omega_P}e.$$

Then, we derive

$$\begin{aligned}\rho_P(e)=\rho_P(e-c)&=\int_{\omega_P}fw+\sum_{E\in\mathcal{E}_P}\int_E j_Ew\\&=\int_{\omega_P}f\Pi_Pw+\sum_{E\in\mathcal{E}_P}\int_E j_E\Pi_Pw+\int_{\omega_P}f(w-\Pi_Pw)\\&=\rho_{\mathcal{S}}(\Pi_Pw)+\int_{\omega_P}(f-\bar f_P)(w-\Pi_Pw)\\&\le\sum_{E\in\mathcal{E}_P}|\alpha_E(w)|\eta_E\|\phi_E\|+\|f-\bar f_P\|_{0,\omega_P}\|w-\Pi_Pw\|_{0,\omega_P}\end{aligned} \tag{3.28}$$

from (3.19), the definition (3.20) of Π_P, the fact that $j_E\in\mathbb{R}$ is constant, the definition (2.15) of η_E, and the Cauchy-Schwarz inequality. Notice that, thanks to the choice of c in the definition of w and $P\in\Omega$, we have

$$\|w\|_{0,\omega_P}\le\|e-c\|_{0,\omega_P}\lesssim h_P\|\nabla e\|_{0,\omega_P} \tag{3.29}$$

by a Poincaré inequality, cf., e.g., [5]. Utilizing (3.23), $\|\phi_P\|_{\infty,\omega_P}\le 1$, $\|\nabla\phi_P\|_{\infty,\omega_P}\lesssim h_P^{-1}$, and (3.29), we obtain

$$\begin{aligned}|\alpha_E(w)|&\lesssim h_P^{-1}\big\{\|w\|_{0,\omega_P}+h_P\|\nabla w\|_{0,\omega_P}\big\}\\&\lesssim h_P^{-1}\Big\{\|(e-c)\phi_P\|_{0,\omega_P}+h_P\|\nabla\big((e-c)\phi_P\big)\|_{0,\omega_P}\Big\}\\&\lesssim\|\nabla e\|_{0,\omega_P}\approx\|\phi_E\|^{-1}\|\nabla e\|_{0,\omega_P}\end{aligned} \tag{3.30}$$

for all $E\in\mathcal{E}_P$. In a similar way, we get

$$\|w-\Pi_Pw\|_{0,\omega_P}\lesssim\|w\|_{0,\omega_P}+h_P\|\nabla w\|_{0,\omega_P}\lesssim h_P\|\nabla e\|_{0,\omega_P} \tag{3.31}$$

using (3.24). The desired estimate (3.27) follows by inserting these two inequalities into (3.28).

Case 2: $P \in \mathcal{N} \cap \partial\Omega$. We claim

$$\rho_P(e) \lesssim \Big(\sum_{E \in \mathcal{E}_P} \eta_E + h_P \|f\|_{0,\omega_P} \Big) \|\nabla e\|_{0,\omega_P}. \tag{3.32}$$

To prove (3.32), we again start from the inequality

$$\rho_P(e) \le \sum_{E \in \mathcal{E}_P} |\alpha_E(w)| \eta_E \|\phi_E\| + \|f\|_{0,\omega_P} \|w - \Pi_P w\|_{0,\omega_P}, \tag{3.33}$$

where this time we set

$$w = (e - c)\phi_P, \qquad c = 0.$$

There is no freedom in the choice of c, since $P \notin \Omega$ so that we cannot invoke (3.19). However, e vanishes at least on one edge of $\partial\omega_P$, because $P \in \partial\Omega$. Hence, the generalized Poincaré-Friedrichs inequality [4, Lemma 3.4] can be applied again to obtain (3.30) and (3.31). Inserting these inequalities into (3.33), we get the desired bound.

To conclude the proof, we sum the estimates for the two cases over $P \in \mathcal{N}$, invoke the definition of the oscillation term, and apply the Cauchy-Schwarz inequality, to obtain

$$\|\nabla e\|_{0,\Omega}^2 = \|e\|^2 = \rho_{\mathcal{S}}(e) \lesssim \Big(\sum_{E \in \mathcal{E}} \eta_E^2 + \mathrm{osc}(\mathcal{T}, f)^2 \Big)^{\frac{1}{2}} \|\nabla e\|_{0,\Omega}.$$

This concludes the proof. □

4. Small data oscillation implies the saturation assumption

The following simple corollary of Proposition 1.1 and Theorem 3.2 was not noticed by Bornemann et al. [1] but could have been used as a shortcut to the arguments of Dörfler and Nochetto [3].

Proposition 4.1. *There is a $\mu > 0$ such that small data oscillation*

$$osc(\mathcal{T}, f) \le \mu \|u - u_{\mathcal{S}}\| \tag{4.34}$$

implies the saturation assumption (1.4).

Proof. Combining Theorem 3.2 with (2.12), we obtain

$$\|u - u_{\mathcal{S}}\|^2 \le C(\|u_{\mathcal{Q}} - u_{\mathcal{S}}\|^2 + osc(\mathcal{T}, f)^2) \tag{4.35}$$

with some constant C. Inserting (4.34) into (4.35) with $\mu^2 < 1/C$, we get (1.5) with some $\tilde{C} > 0$. Then (1.4) follows from Prosition 1.1. □

5. Local lower bound (efficiency)

As a consequence of (1.3) and the equivalence (2.12), we immediately get the global lower bound

$$\|e_{\mathcal{V}}\|^2 = \sum_{E \in \mathcal{E}} \eta_E^2 \le \|u - u_{\mathcal{S}}\|.$$

We now show that the local error indicators η_E even provide *local* lower bounds of the error.

Theorem 5.1. *Let*

$$\omega_E = \text{int supp } \phi_E, \qquad \|v\|_{0,\omega_E} = (\int_{\omega_E} v^2)^{1/2}.$$

Then

$$|\eta_E| \le \|\nabla(u - u_{\mathcal{S}})\|_{0,\omega_E}, \qquad E \in \mathcal{E}. \tag{5.36}$$

Proof. By definition of η_E and $\rho_{\mathcal{S}}$ and by the Cauchy-Schwarz inequality, we get

$$|\eta_E| = \rho_{\mathcal{S}}(\frac{\phi_E}{\|\phi_E\|}) = \int_{\omega_E} \nabla(u - u_{\mathcal{S}}), \nabla(\frac{\phi_E}{\|\phi_E\|^2})\, dx \le \|\nabla(u - u_{\mathcal{S}})\|_{0,\omega_E}.$$

□

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