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1st exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, May 4th, 2016 (10:15 a.m.)

Exercise 1 (4 TP)
Consider the space $V = \{f \in C(\mathbb{R}) \,|\, \|f\|_\infty < \infty\}$ with norm $\|\cdot\|_\infty$. Let $u_n(x) = \max(0, 1 - |x - n|)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Show that $(u_n)_{n\in\mathbb{N}}$ has no converging subsequence in V and conclude that the unit sphere in V is not compact.

Exercise 2 (4 TP)
Show that the Laplace operator is invariant under rotations of the coordinate system: Let $A \in \mathbb{R}^{d\times d}$ be an orthogonal matrix with columns $a_i \in \mathbb{R}^d$. Given a rotation around a center $c \in \mathbb{R}^d$, defined by $\psi(x) = A(x - c) + c$, prove that the sequence of identities

$$\Delta u(x) = (\Delta(u \circ \psi))(\psi^{-1}x) = \Delta_A u(x) := \sum_{i=1}^{d} \frac{\partial^2}{\partial a_i \partial a_i} u(x) \quad \forall x \in \mathbb{R}^d$$

holds for all $x \in \mathbb{R}^d$.

Exercise 3 (5 TP)
For all $\varepsilon \in \mathbb{R}_{>0}$ let

$$f_\varepsilon(x) = \begin{cases} -1 & , x \in [-1, -\varepsilon] \\ \frac{x}{\varepsilon} & , x \in (-\varepsilon, \varepsilon) \\ 1 & , x \in [\varepsilon, 1] \end{cases} \qquad u_\varepsilon(x) = \begin{cases} -\frac{1}{2}x^2 - \frac{\varepsilon^2-3}{6}x - \frac{\varepsilon^2}{6} & , x \in [-1, -\varepsilon] \\ \frac{1}{6\varepsilon}x^3 + \frac{3\varepsilon-\varepsilon^2-3}{6}x & , x \in (-\varepsilon, \varepsilon) \\ \frac{1}{2}x^2 - \frac{\varepsilon^2-3}{6} + \frac{\varepsilon^2}{6} & , x \in [\varepsilon, 1]. \end{cases}$$

a) Verify that u_ε solves

$$\Delta u_\varepsilon = f_\varepsilon \text{ on } [-1, 1], \qquad u_\varepsilon = 0 \text{ on } \{-1, 1\}$$

in the classical sense.

Please turn over...

b) Prove that u_ε converges pointwise to

$$u(x) = \begin{cases} -\frac{x}{2}\,(x+1) & , \; x \le 0 \\ \frac{x}{2}\,(x-1) & , \; x \ge 0. \end{cases}$$

c) Show that u_ε does not converge to u in $C^2([-1,1])$.

d) Let $\|v\|_{H^1} := \sqrt{\int_{-1}^{1} (v'(x))^2 \,\mathrm{d}x}$. Show that $\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{H^1} = 0$.

e) Let

$$f(x) = \begin{cases} -1 & \text{for } x \le 0 \\ 1 & \text{for } x > 0. \end{cases}$$

Prove that u is a weak solution of

$$\Delta u = f \text{ on } [-1,1], \qquad u = 0 \text{ on } \{-1,1\}$$

by verifying the variational equality

$$\forall v \in C_0^\infty((-1,1))\colon \; -\int_{[-1,1]} u'(x)\, v'(x)\, \mathrm{d}(x) = \int_{[-1,1]} f(x)\, v(x)\, \mathrm{d}(x).$$

Have fun!