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2nd exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, May 11th, 2016 (10:15 a.m.)

Exercise 1 (4 TP)
Let $\Omega \subseteq \mathbb{R}^d$ be a Green domain. Prove the following statements:

a) If $u \in C^1(\overline{\Omega})$ and $v \in C^2(\overline{\Omega})$ then

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x = -\int_\Omega u(x)\,\Delta v(x)\,\mathrm{d}(x) + \int_{\partial\Omega} u(x)\,\left(\nabla v(x) \cdot n(x)\right)\,\mathrm{d}o(x).$$

b) If $u, v \in C^2(\overline{\Omega})$ then

$$\int_\Omega \left(u(x)\,\Delta v(x) - \Delta u(x)\,v(x)\right)\,\mathrm{d}x = \int_{\partial\Omega} \left(u(x)\nabla v(x) - v(x)\nabla u(x)\right) \cdot n(x)\,\mathrm{d}o(x).$$

Exercise 2 (4 TP)
Let $\Omega \subseteq \mathbb{R}^n$ such that $\partial\Omega$ is smooth. Define

$$H_C := \left\{v \in C^1(\overline{\Omega}) \mid v|_{\partial\Omega} = 0\right\}$$

and suppose $f \in H_C$. Show that there exists at most one $u \in H_C$ such that the variational equality

$$\forall v \in H_C\colon\ \int_\Omega \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x = \int_\Omega f(x)\,v(x)\,\mathrm{d}x$$

holds.

Please turn over...

Exercise 3 (4 TP)
Let $\Omega = (-1, 1)$ and $\|u\|_{H^1(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + \|u'\|^2_{L^2(\Omega)}}$ for all $u \in C^1(\Omega)$. Show that

a) the space $C^0(\Omega)$ is not complete with respect to $\|\cdot\|_{L^2(\Omega)}$ and

b) the space $C^1(\Omega)$ is not complete with respect to $\|\cdot\|_{H^1(\Omega)}$.

Exercise 4 (4 TP)
The (nonlinear) Cauchy-Green strain tensor and the linear strain tensor for a smooth displacement vector field $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ are given by

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{l=1}^{d} \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right), \qquad \mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Explain why the resulting displacement of a rigid body motion applied after u is given by $\tilde{u}(x) = A(x + u(x)) - x + c$ for an orthogonal matrix A and a vector c. Show that E is invariant under any rigid body motion while $\mathcal{E}$ is only invariant under translations but not under rotations.

Have fun!