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3rd exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, May 18th, 2016 (10:15 a.m.)

Exercise 1 (8 TP)
Let $\Omega \subseteq \mathbb{R}^n$ with smooth boundary. Furthermore, suppose that $\gamma\colon \mathbb{R}^n \to \mathbb{R}$ is strictly convex and define $J(u) := \int_\Omega \gamma(\nabla u(x))\,\mathrm{d}x$ for all $u \in C^1(\overline{\Omega})$.

a) For each $f \in C^1(\overline{\Omega})$ define $V_f := \{u \in C^1(\overline{\Omega}) \mid u|_{\partial\Omega} = f|_{\partial\Omega}\}$. Show that J is strictly convex on the set V_f for any $f \in C^1(\overline{\Omega})$.

b) Show that there can be at most one $u^* \in V_f$ such that $J(u^*) = \inf_{u \in V_f} J(u)$.

c) From now on assume that the minimizer $u^* \in V_f$ of J over V_f exists. Show that

$$\forall v \in V_0\colon \quad \int_\Omega \nabla\gamma(\nabla u^*(x)) \cdot \nabla v(x)\,\mathrm{d}x = 0.$$

d) Assume furthermore that $\gamma(x) = \sqrt{1 + \|x\|^2}$ for all $x \in \mathbb{R}^n$ and that $u^* \in C^2(\overline{\Omega})$. Prove that for every $x \in \Omega$

$$\operatorname{div}\left(\frac{\nabla u^*(x)}{\sqrt{1 + \|\nabla u^*(x)\|^2}}\right) = 0.$$

Exercise 2 (8 TP)

a) Suppose a map $A \in C(\mathbb{R}^n, \mathbb{R}^{n\times n})$, a vector field $b \in C(\mathbb{R}^n, \mathbb{R}^n)$ and a scalar function $c \in C(\mathbb{R}^n, \mathbb{R})$. The differential operator defined by

$$L(x,u) := A(x) : D^2 u(x) + b(x) \cdot \nabla u(x) + c(x)u(x)$$

is called *elliptic in* $x \in \mathbb{R}^n$ if and only if all eigenvalues of $A(x)$ are unequal 0 and have the same sign.

Let $n = 2$. Show that L is elliptic in $x \in \mathbb{R}^n$ in the sense of (a) if and only if $A_{11}(x)A_{22}(x) - A_{12}(x)A_{21}(x) > 0$.

Please turn over...

b) Let $m \in \mathbb{N}$ and $a_\alpha \in C(\mathbb{R}^n, \mathbb{R})$ for every multi-index α with $|\alpha| \leq m$. The differential operator defined by

$$L(x,u) := \sum_{|\alpha| \leq m} a_\alpha(x) \partial_\alpha u(x)$$

is called *elliptic in* $x \in \mathbb{R}^n$ if and only if for all $\xi \in \mathbb{R}^n \setminus \{0\}$

$$L'(x,\xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha \neq 0.$$

Show that every elliptic differential operator in the sense of (a) is elliptic in the sense of (b).

c) Suppose that L is an elliptic differential operator in the sense of (b). Prove that m is even.

Remarks:

- For two matrices $A, B \in \mathbb{R}^{n \times n}$ we define $A : B := \sum_{i,j=1}^n A_{ij} B_{ij}$ to be the sum of the element-wise products of A and B.
- An operator in the sense of (a) is called *parabolic* if A has one zero-eigenvalue and all other eigenvalues are unequal 0 and have the same sign. The operator L is called *hyperbolic* if A has one strictly negative (or positive) eigenvalue and all other eigenvalues are strictly positive (or negative).
- For a vector $\xi \in \mathbb{R}^n$ and a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ we define $\xi^\alpha := \prod_{i=1}^n \xi_i^{\alpha_i}$.
- The last statement is even true for complex coefficients if $n \geq 3$.

Exercise 3 (4 extra-TP)
Classify the following PDEs.

a) $-\text{div}(\alpha(x)\nabla u) = 0, \qquad \alpha \in C^1(\Omega), \quad \alpha(x) \geq \alpha_0 > 0.$

b) $\varepsilon \Delta u - \vec{\beta} \nabla u = 0, \qquad$ (i) for $\varepsilon \neq 0$, (ii) for $\varepsilon = 0$.

c) $(c_0^2 - u_x^2) u_{xx} - 2 u_x u_y u_{xy} + (c_0^2 - u_y^2) u_{yy} = 0 \qquad c_0 > 0.$

Have fun!