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4th exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, May 25th, 2016 (10:15 a.m.)

Exercise 1 (6 TP)
Let $n \in \mathbb{N}_{\geq 2}$, $a \in \mathbb{R}^n$ and define for all $x \in \mathbb{R}^n \setminus \{a\}$

$$s(x,a) = \begin{cases} -\frac{1}{2\pi}\log(\|x-a\|) & \text{for } n = 2, \\ -\frac{1}{(2-n)\omega_n}\|x-a\|^{2-n} & \text{for } n > 2. \end{cases}$$

Here $\omega_n := \int_{\partial B_1^n(0)} \mathrm{d}\sigma$ is the surface area of the unit sphere in $\mathbb{R}^n$. In the following n denotes the *inward* pointing unit normal of $B_r^n(a)$.
Prove the following statements.

a) It holds

$$\lim_{r \searrow 0} \int_{\partial B_r^n(a)} s(x,a)\,\mathrm{d}\sigma = 0.$$

b) For each $r \in \mathbb{R}_{>0}$

$$\int_{\partial B_r^n(a)} \partial_n s(x,a)\,\mathrm{d}\sigma = 1.$$

c) Suppose that $v \in C^1(\mathbb{R}^n)$. Then

$$\lim_{r \searrow 0} \int_{\partial B_r^n(a)} v(x)\partial_n s(x,a)\,\mathrm{d}\sigma = v(a).$$

Hint: Use the uniform continuity of v over the relevant sets.

Please turn over...

Exercise 2 (6 TP)

a) Let $a \in \mathbb{R}$. Specify the set of all functions $s \in C^2(\mathbb{R}^+)$ such that $s(|\cdot - a|) \in C^2(\mathbb{R} \setminus \{a\})$ is harmonic.

b) For any $a \in \mathbb{R}$ find the one-dimensional singularity function at a, i. e., the harmonic function $s(\cdot, a) \in C^2(\mathbb{R} \setminus \{0\})$ such that

$$\lim_{r \searrow 0} \int_{\partial B_r^1(a)} s(x, a) \, \mathrm{d}\sigma = 0,$$

and for all $r \in \mathbb{R}^+$

$$\int_{\partial B_r^1(a)} \partial_n s(x, a) \, \mathrm{d}\sigma = 1.$$

Here $B_r^1(a) = (a - r, a + r)$ and n is the *inward pointing* unit normal.

c) Let $\Omega \subseteq \mathbb{R}$ be a bounded domain. Find the Green's function G on Ω, i. e. find for every $a \in \Omega$ a harmonic function $\Phi(\cdot, a) \in C^2(\overline{\Omega})$ such that the function $G(\cdot, a) := s(\cdot, a) + \Phi(\cdot, a)$ fulfills $G(x, a) = 0$ for each $x \in \partial\Omega$.

Exercise 3 (4 TP)

a) Define the transformation map from polar coordinates to Cartesian coordinates by

$$X \colon \mathbb{R}_0^+ \times \mathbb{R} \longrightarrow \mathbb{R}^2, \qquad (r, \varphi) \longmapsto r \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}.$$

For $u \in C^2(\mathbb{R}^2 \setminus \{0\})$ and $\widehat{u} = u \circ X \in C^2(\mathbb{R}^+ \times \mathbb{R})$ show that

$$(\Delta u)(X(r, \varphi)) = \partial_{rr}\widehat{u}(r, \varphi) + \frac{1}{r}\partial_r \widehat{u}(r, \varphi) + \frac{1}{r^2}\partial_{\varphi\varphi}\widehat{u}(r, \varphi) \qquad \forall (r, \varphi) \in \mathbb{R}^+ \times \mathbb{R}.$$

b) Let $u(x) = \log|x|$. Show that u is harmonic on $\mathbb{R}^2 \setminus \{0\}$.

Have fun!