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5th exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, June 1st, 2016 (10:15 a.m.)

Exercise 1 (8 PP)
Let $\Omega := [0,1]^2$ and $f, g \in C(\overline{\Omega})$.

a) For each $N \in \mathbb{N}$ let $h := \frac{1}{N}$ and $\Omega_N := \{(ih, jh) \mid (i,j) \in \{0,\dots,N\}^2\}$. Implement the Shortley-Weller method for the problem

$$\begin{aligned} -\Delta_h U(x) &= f(x) \text{ for each } x \in \Omega_N \\ U(x) &= g(x) \text{ for each } x \in \partial\Omega_N \end{aligned} \tag{1}$$

by writing functions

```
A = OperatorAssembler( N )
F = FunctionalAssembler( f, g, N )
```

which assemble the coefficient matrix A and the right-hand side vector F of this discrete equation for given function handles `f` and `g`. The returned matrix should be stored in a sparse format.

b) Let

$$\begin{aligned} f_1 &= 2(x_1^2 + x_2^2 - 2) & g_1 &= 0 & u_1 &= (x_1^2 - 1)(x_2^2 - 1) \\ f_2 &= -4 & g_2 &= \|x\|^2 & u_2 &= \|x\|^2 \\ f_3 &= (2\pi^2)^{-1}\sin(\pi x_1)\sin(\pi x_2) & g_3 &= 0 & u_3 &= \sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

Solve the discrete problem (1) given (f_k, g_k) and $N = 2^l$ for all $k \in \{1,2,3\}$ and $l \in \{2,\dots,8\}$. Plot the graphs of the discrete solutions U. In addition, plot the errors $\max_{x\in\Omega_N} |u_k(x) - U_k(x)|$ versus $h = \frac{1}{N}$ in a logarithmic scale with 1:1 aspect-ratio. You can use the command `axis equal` and the function `loglog`.

Remark: It is necessary that you add comments to your code which explain your implementation.

Please turn over...

Exercise 2 (4TP)
Suppose that $k \in \{3, 4\}$ and $h \in \mathbb{R}_{>0}$.

a) Let $I \subseteq \mathbb{R}$ be an interval and $u \in C^k(\overline{I})$. Show that there exists a constant $c \in \mathbb{R}_{>0}$ such that for all $x \in I$ with $[x-h, x+h] \subseteq I$ holds

$$|\Delta u(x) - \Delta_h u(x)| \leq c h^{k-2}.$$

b) Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain and $u \in C^k(\overline{\Omega})$. Prove that there exists a constant $c \in \mathbb{R}_{>0}$ such that for all $x \in \Omega$ with $B_h(x) \subseteq \Omega$ holds

$$|\Delta u(x) - \Delta_h u(x)| \leq c h^{k-2}.$$

Remark: We define for every $n \in \mathbb{N}$ and $u \in C(\mathbb{R}^n)$

$$\Delta_h u(x) := \frac{1}{h^2} \sum_{i=1}^{n} \left(u(x - he_i) - 2u(x) + u(x + he_i) \right).$$

Exercise 3 (4 TP)
Let $[a, b] \subseteq \mathbb{R}$ be an interval. Suppose $f \in C(\overline{I})$ and that $u \in C^2(\overline{I})$ is a solution of

$$-\Delta u = f, \qquad u|_{\{a,b\}} = 0.$$

Furthermore, let $N \in \mathbb{N}$ and $h := \frac{b-a}{N}$. For each $i \in \{0, \ldots, N\}$ define $x_i := a + ih$ and for every $i \in \{1, \ldots, N-1\}$ let

$$\lambda_i \colon \overline{I} \longrightarrow \mathbb{R}, \qquad x \longmapsto \begin{cases} \frac{1}{h}(x - x_{i-1}) & \text{for } x \in [x_{i-1}, x_i] \\ -\frac{1}{h}(x - x_{i+1}) & \text{for } x \in [x_i, x_{i+1}] \\ 0 & \text{for } x \notin [x_{i-1}, x_{i+1}]. \end{cases}$$

Prove for all $i \in \{1, \ldots, N-1\}$ that

$$-\Delta_h u(x_i) := -\frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{h^2} = \frac{\int_I f(x)\lambda_i(x)\,\mathrm{d}x}{\int \lambda_i(x)\,\mathrm{d}x} =: f_h(x_i).$$

Hint: Multiply the PDE by λ_i, then integrate and use partial integration.

Have fun!