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6th exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, June 8th, 2016 (10:15 a.m.)

Exercise 1 (8 PP)
Implement the Shortley-Weller method for the problem

$$\begin{aligned} -\Delta_h U(x) &= f(x) \text{ for each } x \in \Omega_N \\ U(x) &= g(x) \text{ for each } x \in \partial\Omega_N \end{aligned} \tag{1}$$

for arbitrary bounded domains $\Omega \subseteq \mathbb{R}^2$ that are given by $\{x \in \mathbb{R}^2 \mid \gamma(x) \geq 0\}$ for some $\gamma \in C(\mathbb{R}^2, \mathbb{R})$.
Write functions

```
A = OperatorAssembler( V, E )
F = FunctionalAssembler( f, g, V, E )
```

which assemble the coefficient matrix A and the right-hand side vector F of the discrete equation for given function handles `f`, `g` and a list of all grid nodes L with a matrix of neighbors E. The returned matrix should be stored in a sparse format.
Let

$$\begin{aligned} \gamma_1(x) &= \frac{1}{4} - (x_1^2 + x_2^2) \\ \gamma_2(x) &= (1 - \max(\operatorname{sign}(x_1), 0) \max(\operatorname{sign}(-x_2), 0)) - (x_1^2 + x_2^2) \end{aligned}$$

and

$$f_1(x) = 1, \qquad g_1(x) = 0, \qquad f_2(r, \varphi) = 0, \qquad g_2(r, \varphi) = \sin\left(\frac{2}{3}\varphi\right).$$

Solve the discrete problem (1) given (γ_k, f_k, g_k) and $h = 2^{-l}$ for all $k \in \{1, 2\}$ and $l \in \{2, \dots, 8\}$. Plot the graphs of the discrete solutions U. In addition, find the exact solutions u_k and plot the errors $\max_{x\in\Omega_N} |u_k(x) - U_k(x)|$ versus $h = \frac{1}{N}$ in a logarithmic

Please turn over...

scale with 1:1 aspect-ratio. You can use the command `axis equal` and the function `loglog`.

Remark: You may use the function `createFDGrid` from the website that creates a list of grid points with a neighbor matrix for a given function handle γ. Do not forget that it is necessary that you add comments to your code which explain your implementation.

Exercise 2 (4TP)
Let $A \in \mathbb{R}^{n \times n}$ be the coefficient matrix from the Shortley-Weller method. Show that its inverse $B := A^{-1}$ fulfills $B_{ij} > 0$ for all $i, j \in \{1, \ldots, n\}$.

Hint: You may use the discrete strong maximum principle: Let U be discretely subharmonic. If there exists $x \in \Omega_h^\circ$ such that $U(x) = \max_{y \in \partial\Omega_h} U(y)$, then U is constant.

Exercise 3 (4TP)
Let us consider the space ℓ^2 of all real sequences $x = (x_n)$, which are quadratically summable. This space is equipped with the norm $\|x\|_2 = (\sum_n^\infty x_n^2)^{1/2}$. Check that the bilinear form given by

$$a(x, y) := \sum_{n=1}^{\infty} 2^{-n} x_n y_n$$

is positive definite and continuous, but not ℓ^2-elliptic. Furthermore, show that, given $l(x) = \sum_{n=1}^\infty 2^{-n} x_n$, a continuous linear functional is defined on ℓ^2, for which $J : x \mapsto \frac{1}{2}a(x, x) - l(x)$ does not attain its minimum in ℓ^2.

Have fun!