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8th exercise for the lecture

Numerics III

Summer Term 2016

`http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php`

Due: Wednesday, June 29th, 2016 (10:15 a.m.)

Exercise 1 (4 TP)
Let $\Omega \subset \mathbb{R}^n$ be a domain with a sufficiently smooth boundary. Consider the boundary value problem

$$\begin{aligned} -\alpha \Delta u + \beta \cdot \nabla u = f \quad & \text{in } \Omega \\ u = 0 \quad & \text{on } \partial\Omega \end{aligned}$$

with $\alpha > 0$, $\beta \in \mathbb{R}^n$ and a given function $f \in C(\overline{\Omega})$.

a) Derive a weak formulation of this problem and show, that the resulting variational problem is well posed in the Sobolev space $H_0^1(\Omega)$.

b) How does the variational problem change, if β is a function in $C^1(\overline{\Omega})^n$, and under which condition do we have still a well posed problem?

Exercise 2 (4 TP)
Let $\Omega = [0,1]^2$, $\partial_N = (0,1) \times \{0\}$, $\partial_D = \partial_\Omega \setminus \partial_N$, $f \in L^2(\Omega)$, $g_D, g_N \in C(\overline{\Omega})$ and $X := \{v \in H^1(\Omega) \mid \operatorname{tr}_{\partial_D} v = g_D\}$. Furthermore assume that there exists a $u \in X \cap C^2(\overline{\Omega})$ such that

$$\forall v \in X\colon\ \langle \nabla u, \nabla(v - g_D)\rangle_{L^2(\Omega)} = \langle f, v - g_D\rangle_{L^2(\Omega)} + \langle g_N, v - g_D\rangle_{L^2(\partial_N)}.$$

Show that u then is a classical solution of

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g_D \text{ on } \partial_D \\ \frac{\partial u}{\partial n} &= g_N \text{ on } \partial_N. \end{aligned}$$

Hint: Show first that $-\Delta u = f$ holds pointwise by choosing appropriate test functions.

Please turn over...

Exercise 3 (2 TP + 6 PP)
Let $\Omega = [-1, 1]$ and $P_0^k = \{v \in P^k \mid v(-1) = 0 = v(1)\}$ and $T^k = \text{span}_{j\in\{1,\dots,k\}} \sin(j\pi x)$.

a) Show for $k \geq 2$ that $P_0^k = \psi P^{k-2}$ where $\psi = x^2 - 1$.

b) Let $S \subseteq H_0^1(\Omega)$ be an n-dimensional subspace with basis $\{\varphi_1, \dots, \varphi_n\}$. Write functions

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A = OperatorAssembler( basis )
F = FunctionalAssembler( basis, f )
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where `basis` is an appropriate list of function handles and `f` is a function handle. The output is given by the matrix $A \in \mathbb{R}^{n\times n}$ with $A_{ij} = (\nabla\varphi_i, \nabla\varphi_j)_{L^2(\Omega)}$ and by the vector $F \in \mathbb{R}^n$ with $F_i = (f, \varphi_i)_{L^2(\Omega)}$.

Remark: You may use Matlab's `integral` function.

c) Let $k \in \{1, \dots, 7\}$ and $S \in \{P_0^{k+1}, T^k\}$. Compute the weak solution of

$$-\Delta u = f, \qquad u|_{\partial\Omega} = 0$$

over S for $f \in \{1, \sin(\pi\cdot), \text{sign}\}$. Plot the resulting functions and compute the errors to the exact solution in the H^1-seminorm.

Have fun!