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9th exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, July 6th, 2016 (10:15 a.m.)

Exercise 1 (4 TP)
Let $\Omega \in \mathbb{R}^2$ be a domain with polygonal boundary and let $\mathcal{T}$ be a triangulation of Ω. Show that $S^{(m)}$ is a closed subspace of $H^1(\Omega)$.

Exercise 2 (4 TP)
Consider the grid $a = x_0 < x_1 < \ldots < x_n = b$ on the interval (a, b). Let u be the solution of

$$u \in H_0^1(a,b) : \qquad (u', v') = (f, v) \qquad \forall v \in H_0^1(a,b)$$

and let $u_h \in S_h$ be the approximation in the linear finite element space $S_h \subset H_0^1(a,b)$ on the above grid. Show that u_h coincides with the linear interpolation of u on the grid, i. e. $u_h(x_i) = u(x_i)$ for $i = 0, \ldots, n$.

Please turn over...

Exercise 3 (4 Extra TP)
In the lecture, an approximation of the variational problem

$$u \in H_0^1(\Omega): \quad a(u,v) = l(v) \quad \forall v \in H_0^1(\Omega)$$

was derived by using the finite element space S_h. The resulting variational problem

$$u_h \in S_h: \quad a(u_h,v) = l(v) \quad \forall v \in S_h$$

is rewritten as the linear system of equations

$$AU = b\ .$$

Show that A is positive definite if $a(\cdot,\cdot)$ is elliptic and that symmetry of $a(\cdot,\cdot)$ implies symmetry of A.

Exercise 4 (4 Extra TP)
Let $\mathcal{T}_0 = \{[0,1/2];[1/2,1]\}$ be a partition of the interval $[0,1]$ and $j \in \mathbb{N}$. One can inductively define partitions $\mathcal{T}_k$ for $k = 1,\ldots,j$ by successively bisecting the sub-intervals of $\mathcal{T}_{k-1}$. We define the set of all inner grid points on the level k by $\mathcal{N}_k$. Furthermore, let S_k be the space of the continuous functions with zero boundary conditions which are piecewise linear over $\mathcal{T}_k$. Finally, for each $k \in \{0,\ldots,j\}$ and $p \in \mathcal{N}_k$, let $\lambda_p^{(k)}$ be the nodal basis function corresponding to the point p on level k. Then we define the *hierarchical basis* of S_j by

$$H := \left\{\lambda_{1/2}^{(0)}\right\} \cup \left(\bigcup_{k=1}^{j} \{\lambda_p^{(k)} : p \in \mathcal{N}_k \backslash \mathcal{N}_{k-1}\}\right).$$

Draw a sketch of the hierarchical basis and show that H is an a-orthogonal basis of S_j with respect to

$$a(v,w) = \int_0^1 v'(x)w'(x)\,\mathrm{d}x\ .$$

Exercise 5 (8 PP)

a) Make yourself familiar with the MATLAB programmes `basis.m`, `quadrature.m` and `uniform_grid.m` on the homepage.

b) Write a MATLAB programme `A = assemble_P1(grid, local_assem, Q)`, which assembles the global matrix $A_{i,j} = a(\lambda_j, \lambda_i)$ for the linear finite elements nodal basis $\{\lambda_i\}$ on the grid `grid`. The matrix should be calculated as a sum of element matrices assembled by the function `M = local_assem(T, B, Q)`. Thereby the columns of `T` give a triangle of the grid, `B = basis(1)` the local basis and `Q` a quadrature rule defined on the unit simplex. Test your programme by assembling the stiffness and the mass matrix for a uniform grid, using the local assemblers `assemble_stiff` and `assemble_mass` and appropriate quadrature rules.

c) Write a MATLAB programme `[A,b] = assemble_dirichlet(grid, A, b, g)`, which „includes“Dirichlet boundary conditions, given by the function `function y = g(x)`, in the matrix `A` and the right-hand side `b`.

d) Use your programmes to approximate a solution of the problem

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega$$

for

$$f(x) = \begin{cases} 0.2 & \text{for } |x - (0.5, 0.5)| \leq 0.2 \\ 0 & \text{else} \end{cases}$$

and $g = 0$ with linear finite elements on the unit square $\Omega = [0,1]^2$ and on the unit circle $\Omega = K_1(0)$, and visualize the solution with the MATLAB command `trisurf`.

Advices:

- You can load a grid on the unit circle with the command `grid = load('circle')`, using the file 'circle' on the homepage.
- The right-hand side b can be assembled by linear interpolation of f, i.e. by evaluation at the grid points and multiplication with the mass matrix.

Have fun!