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10th exercise for the lecture

Numerics III

Summer Term 2016

http://numerik.mi.fu-berlin.de/wiki/SS_2016/NumericsIII.php

Due: Wednesday, July 13th, 2016 (10:15 a.m.)

Exercise 1 (8 TP)

a) Consider $F_\tau = B_\tau + p_0 : T \to \tau$ with B_τ linear, the transformation of the unit triangle $T = \operatorname{conv}\{0, e_1, e_2\}$ on a triangle $\tau = \operatorname{conv}\{p_0, p_1, p_2\} \subset \mathbb{R}^2$. Show the estimates

$$|v|_{1,\tau} \le c_2 |B_\tau^{-1}| \, |\det B_\tau|^{\frac{1}{2}} \, |v \circ F_\tau|_{1,T},$$
$$|v \circ F_\tau|_{2,T} \le c_1 |B_\tau|^2 \, |\det B_\tau|^{-\frac{1}{2}} \, |v|_{2,\tau}$$

by using the chain rule for $v \in H^2(\tau)$.

b) Consider $F = B_\tau + p_0$ as defined in a). Show that

$$|B_\tau| \le \frac{\sqrt{2}}{2} r_\tau, \qquad |B_\tau^{-1}| \le \frac{\sqrt{2}}{4} \rho_\tau^{-1}$$

with r_τ and ρ_τ the radii of the outer and inner circle of τ.

c) Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be a family of triangulations of a domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary. Prove the following error estimate for the interpolation operator $I_h : C(\Omega) \to S_h$

$$\|u - I_h u\|_{1,\Omega} \le \left(c \max_{\tau \in \mathcal{T}_h} \frac{r_\tau}{\rho_\tau} \right) h |u|_2 \qquad \forall u \in C^2(\Omega).$$

You can use the estimate

$$\|v - I_T v\|_{1,T} \le c |v|_{2,T}$$

for functions v and the interpolation I_T on the unit triangle.

Please turn over...

Exercise 2 (4 extra TP + 6 extra TP)
Consider the variational equality

$$u \in H_0^1(\Omega) \qquad a(u,v) = l(v) \qquad \forall v \in H_0^1(\Omega)$$

with a H_0^1-elliptic symmetric bilinearform a, $l \in (H_0^1(\Omega))'$ and the finite element solutions $u_{\mathcal{S}} \in S^{(1)}$ and $u_{\mathcal{Q}} \in S^{(2)}$. Assume the saturation assumption

$$\exists \beta \in (0,1) \quad : \quad \|u - u_{\mathcal{Q}}\|_a \leq \beta \|u - u_{\mathcal{S}}\|_a \tag{1}$$

is fullfilled.

a) Show the following estimate for the discretezation error:

$$\sqrt{1-\beta^2}\|u - u_{\mathcal{S}}\|_a \leq \|u_{\mathcal{S}} - u_{\mathcal{Q}}\|_a \leq \|u - u_{\mathcal{S}}\|_a.$$

b) Write a MATLAB programme `A = assemble_P2(grid, local_assem, Q)`, which assembles the stiffness matrix and the mass matrix for quadratic finite elements, using the corresponding local assemblers and appropriate quadrature rules. Note that there are dgrees of freedom on the edges.

c) Use your programme and programmes from the homepage to calculate the a posteriori error estimate $\|u_{\mathcal{S}} - u_{\mathcal{Q}}\|_a$ for the solution of

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

with

$$f(x) = \begin{cases} 0.2 & \text{for } |x - (0.5, 0.5)| \leq 0.2, \\ 0 & \text{else,} \end{cases}$$

on an uniform grid $\mathcal{T}_h$ on $\Omega = (0,1)^2$ with different h. Use the hierarchical P^2 basis here, <u>not</u> the Lagrange basis. Plot the error $\|u_{\mathcal{S}} - u_{\mathcal{Q}}\|_a$ over h in a suitable scale and interprete your results on the background of the a priori error estimatesfrom the lecture.

d) Define the bilinearform

$$b(u,v) = \sum_{E \in \mathcal{E}} u_{x_E} v_{x_E} a(\lambda_E, \lambda_E)$$

in the space $\mathcal{V} = \text{span}\{\lambda_E | E \in \mathcal{E}\}$, where $\mathcal{E}$ is the set of inner edges and λ_E the P^2 basis function associated to midpoint x_E of the edge $E \in \mathcal{E}$. Why can $\tilde{u}_{\mathcal{Q}} = u_{\mathcal{S}} + d$ with

$$d \in \mathcal{V}: \qquad b(d,v) = l(v) - a(u_{\mathcal{S}}, v) \quad \forall v \in \mathcal{V}$$

be interpreted as an inexact evaluation of $u_{\mathcal{Q}}$? Use your programme calculate and plot the error estimate $\|u_{\mathcal{S}} - \tilde{u}_{\mathcal{Q}}\|_a$ analogue to part c). Compare and interprete your results.

Exercise 3 (4 Extra TP)
Consider the smoothing property

$$\langle A_k v, v\rangle \leq \omega_0 \langle B_k v, v\rangle \qquad \forall v \in \mathbb{R}^{n_k} \tag{2}$$

for symmetric positive definite matrices $A_k, B_k \in \mathbb{R}^{n_k}$.

a) Show that the smoothing property implies

$$\lambda_{\max}(B_k^{-1} A_k) \leq \omega_0.$$

b) Show that the sequence u_k^ν generated by

$$B_k(u_k^{\nu+1} - u_k^\nu) = b_k - A_k u_k^\nu$$

converges to the solution u_k of $A_k u_k = b_k$ if (2) holds with $\omega_0 < 2$.

Have fun!