Institut für Mathematik
Freie Universität Berlin
Prof. Dr. C. Gräser
M. Sc. T. Kies

Numerics III
SS 2016

Name: ______________________________ Matr.-Nr.: ____________________

Course of studies: ◯ Mathematics ◯ Bioinformatics ◯ BMS ◯ Computer Science
◯ other:

Intended degree: ◯ Diplom ◯ Lehramt (Staatsexamen) ◯ other:
◯ Bachelor (Mono) ◯ Bachelor (Kombi, Lehramt) ◯ Master

Note your name on all sheets you hand in and staple them together. Please do not use a pencil. You are allowed to use all your written documents, books, and a non-programmable calculator. Other electronic devices are not allowed. The exam consists of 4 pages on two sheets.

If you want to find your results on the lecture web page next to your matriculation number sign the following declaration:

I agree with the publication of my results next to my matriculation number on the lecture web page.

______________________________ (sign here)

Problem	I	II.1	II.2	$\sum$
Points				

Good luck!

Part I (8 points)

For each of the following statements check if it is 'true' or 'false', note your answer, and explain it by one sentence or a counterexample.

You get one point for each statement where your answer and explanation is correct. If an answer is not correct or no correct explanation is given, you get zero points for the corresponding statement.

a) The Laplace operator is invariant under rotations.

b) The linear strain tensor $\mathcal{E}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is invariant under rotations.

c) Let $\Omega \subseteq \mathbb{R}^2$. Then

$$\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1 + \|\nabla u(x)\|^2}}\right) = 0 \text{ for } x \in \Omega$$

defines a semi-linear elliptic partial differential equation.

d) Define $f(x) = |x|x$ for all $x \in [-1, 1]$. Then $f \in H^2([-1, 1])$.

e) Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain, $\ell \in (H_0^1(\Omega))'$, and $V \subseteq H_0^1(\Omega)$ a closed linear subspace. If $u \in H_0^1(\Omega)$ and $u_V \in V$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)} = \ell(v) \qquad \forall v \in H_0^1(\Omega),$$
$$(\nabla u_V, \nabla v)_{L^2(\Omega)} = \ell(v) \qquad \forall v \in V,$$

then $\|\nabla(u - u_V)\|_{L^2(\Omega)} = \inf_{v \in V} \|\nabla(u - v)\|_{L^2(\Omega)}$.

f) Let $A_h \in \mathbb{R}^{n_h \times n_h}$ the stiffness matrix for a Galerkin discretization of the Laplace operator in $H_0^1((0, 1))$ with piecewise linear finite elements with respect to the nodal basis on a uniform grid with step size h. Then $\kappa(A_h) \in O(h^2)$ for $h \to 0$.

g) Let $\Omega \subseteq \mathbb{R}^n$ be bounded with smooth boundary. Then for all $g \in L^2(\partial\Omega)$ there exists a $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (g, \operatorname{Tr} v)_{L^2(\partial\Omega)} \qquad \forall v \in H_0^1(\Omega).$$

h) Let $\Omega \subset \mathbb{R}^2$ a bounded domain and $g \in C^2(\mathbb{R}^2)$. Then any classical solution of

$$\Delta u = 0 \text{ in } \Omega, \qquad u = g \text{ on } \partial\Omega$$

satisfies $u \in C^2(\overline{\Omega})$.

Please turn over!

Part II (16 points)

Complete **all** of the following exercises!

Problem 1 (2+2+2+2 points)
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with polygonal boundary $\partial\Omega$. Let $u \in H$ (with H to be determined below) denote a solution of the boundary value problem

$$\frac{1}{\tau}u - \epsilon\Delta u + \beta \cdot \nabla u = f \qquad \text{in } \Omega \tag{1}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{2}$$

with $\tau, \epsilon > 0$, $\beta \in \mathbb{R}^d$ and $f \in L^2(\Omega)$.

a) Derive a corresponding weak formulation

$$a(u, v) = \ell(v) \qquad \forall v \in H, \tag{3}$$

i. e. derive an appropriate Hilbert space H, a bilinear form $a\colon H \times H \to \mathbb{R}$, and a continuous linear functional $\ell \in H'$. As usual we require that (but you do not need to prove this)

- $a(\cdot, \cdot)$ only involves partial derivatives up to at most first order,
- any solution of (1) and (2) solves (3),
- any smooth enough solution of (3) solves (1) and (2).

b) Show that the bilinear form $b(v, w) = (\beta \cdot \nabla v, w)_{L^2(\Omega)}$ is continuous on $H^1(\Omega)$.

c) Show that the bilinear form $b(\cdot, \cdot)$ from b) is not symmetric on $H^1(\Omega)$ if $\beta \neq 0$.
(*Hint:* Consider $v(x) = \beta \cdot x$.)

d) Show that the bilinear form $b(\cdot, \cdot)$ from b) is positive definite on $H_0^1(\Omega)$.
(*Hint:* Use partial integration.)

Please turn over!

Problem 2 (1+1+2+2+2 points)
We want to discretize a variational problem with the bilinear form $a(v,w) = (\nabla v, \nabla w)_{L^2(0,1)}$. To this end we consider

- a uniform grid $0 = x_0 < x_1 < \cdots < x_n = 1$ with step size $h = 1/n = x_i - x_{i-1}$,
- the first order finite element space on this grid given by

$$S_h = \{v \in C([0,1]) \,|\, v|_{[x_{i-1},x_i]} \in \Pi^1 \text{ for } i = 1,\ldots,n\},$$

- the nodal basis functions $\mu_i \in S_h$ associated with x_i, $i = 0,\ldots,n$,
- a uniform refinement $0 = y_0 < y_1 < \cdots < y_{2n} = 1$ with step size $h/2 = 1/(2n) = y_i - y_{i-1}$,
- the first order finite element space on the refined grid given by

$$S_{h/2} = \{v \in C([0,1]) \,|\, v|_{[y_{i-1},y_i]} \in \Pi^1 \text{ for } i = 1,\ldots,2n\},$$

- the nodal basis functions $\lambda_i \in S_{h/2}$ associated with y_i, $i = 0,\ldots,2n$.

Note that the grid points y_i are given by

$$y_i = \begin{cases} x_{i/2} & \text{for even } i = 0,2,\ldots,2n, \\ \frac{1}{2}(y_{i+1} - y_{i-1}) = \frac{1}{2}\left(x_{\frac{1}{2}(i+1)} - x_{\frac{1}{2}(i-1)}\right) & \text{for odd } i = 1,3,\ldots,2n-1. \end{cases}$$

a) Show that S_h is a subspace of $S_{h/2}$.

b) For even i and j with $i \neq j$ show that λ_i and λ_j are a-orthogonal, i. e., $a(\lambda_i, \lambda_j) = 0$.

c) For even i show that λ_i is a-orthogonal to S_h, i. e., $a(\lambda_i, v) = 0$ for all $v \in S_h$.

d) For $n = h = 1$ show that $\operatorname{span}\{\mu_0, \mu_1, \lambda_1\} = S_{h/2}$.

e) For $n = h = 1$ compute the stiffness matrix $A \in \mathbb{R}^{3\times 3}$ for the bilinear form $a(\cdot,\cdot)$ and the hierarchical basis $\{\mu_0, \mu_1, \lambda_1\}$.

End of the exam