# Ergodic Theory and Transfer Operators
## — Question sheet 1 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

**Exercise 1 (Sequence of sets)**
Show that if $\{A_n\}$ is an increasing sequence of measurable sets with $A_n \uparrow A$, then $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

*Hint: Consider the disjoint differences $A_n \setminus A_{n-1}$.*

**Exercise 2* (Circle rotation: invariant measure)**
Let $T : S^1 \to S^1$ denote the rigid rotation of the circle of unit circumference by angle $\alpha$; that is the map $x \mapsto x + \alpha$. Let $m$ denote the Lebesgue measure on $S^1$, and $\mathcal{B}$ the Borel $\sigma$-algebra .
Show that for every $A \in \mathcal{B}$ one has $m(T^{-1}A) = m(A)$. Proceed as follows:

(a) Show the claim for intervals $I \subset S^1$.

(b) Conclude from (a) that the claim holds for finite unions of intervals $E = \bigcup_{i=1}^n I_i$, $I_i$ interval, $n \in \mathbb{N}$.

(c) Conclude that the claim holds for the algebra $\mathcal{A}$ of subsets of $S^1$ generated by all intervals.

(d) Consider $\mathcal{M} = \{E \in \mathcal{B} \,|\, m(T^{-1}E) = m(E)\}$. Show that $\mathcal{M}$ is a monotone class.

(e) Use the Monotone Class Theorem (cf. Handout 1) to show the claim for any $A \in \mathcal{B}$.

**Exercise 3 (Invariant sets)**
Let $T : X \to X$ some transformation, and let $E \in \mathcal{B}$ be an invariant set, i.e. $T^{-1}(E) = E$. Prove that its complement, $E^c = X \setminus E$, is also an invariant set.

**Exercise 4* (MATLAB: ergodicity vs mixing)**
Consider the two maps on $X = S^1 \times S^1$:

$$T_1(x,y) = \begin{pmatrix} x + \sqrt{2} \\ y + \sqrt{3} \end{pmatrix} \mod 1, \qquad T_2(x,y) = \begin{pmatrix} x + y \\ x + 2y \end{pmatrix} \mod 1\,.$$

(a) Choose a random point $x \in X$, and plot the points $\{T_1^k x\}_{k=0}^{10^4}$ and then $\{T_2^k x\}_{k=0}^{10^4}$. Do the trajectories seem to "get everywhere" for both systems? Do you see a difference in the density of points for the both systems?

(b) Now, choose 1000 random points in $[0, 0.1] \times [0, 0.1]$, we denote this set of points by $E$. Plot $T_i^1(E), T_i^2(E), \ldots, T_i^7(E)$ for $i = 1, 2$. What kind of qualitative difference do you observe?

*Remark: The observed difference is the difference between "ergodic" and "mixing" behavior, and is going to be characterized in the lectures.*

**Exercise 5 (MATLAB: do you trust the results?)**
Let $T : S^1 \to S^1$ denote the circle doubling map: $x \mapsto 2x \mod 1$. Write a MATLAB program to simulate 60 iterates of $T$, i.e. the set $\{T^k x\}_{k=0}^{60}$ for some initial $x \in S^1$. Is the answer as you expected?

**Exercise 6$^*$ (Recurrence for images)**
Consider the sequence of images seen here.[1] It is generated by applying the same permutation over and over to the pixels of an image. Iterating this procedure, after some (possibly many) steps the original image occurs. Explain this behavior by using Poincaré's Recurrence Theorem, in particular, define a suitable mpt.

*Hint: For these images with* $74 \times 74$ *pixels the state space is a finite set consisting of* $(74^2)!$ *elements.*

[1] http://en.wikipedia.org/wiki/Arnold%27s_cat_map#/media/File:Arnold%27s_Cat_Map_animation_(74px,_zoomed,_labelled).gif