Ergodic Theory and Transfer Operators
— Question sheet 2 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1 (Delta measure)
Let $(X, \mathcal{B})$ be a measurable space and define the *delta measure* at $x \in X$ as

$$\delta_x(A) := \begin{cases} 1, & x \in A; \\ 0, & \text{otherwise.} \end{cases} \qquad \text{for all } A \in \mathcal{B}.$$

(a) Show that δ_x is a probability measure.

(b) If x is a *fixed point* of a measurable map $T : X \to X$, i.e. $Tx = x$, show that δ_x is an invariant probability measure for T.

(c) Generalizing (b), if x is a *periodic point of period* $p \in \mathbb{N}$, i.e. $T^p x = x$, show that

$$\mu = \frac{\delta_x + \delta_{Tx} + \ldots + \delta_{T^{p-1}x}}{p}$$

is an invariant probability measure for T.

Exercise 2* (Measurability)
Let $(X, \mathcal{B}, \mu)$ be a measure space, $T : X \to X$ a measurable transformation, and let $E \in \mathcal{B}$ be given. Consider the set $E_0 \subset X$ where E_0 is the set of all states which visit E infinitely often:

$$E_0 := \left\{ x \in X \,\middle|\, T^k(x) \in E \text{ for infinitely many } k \geq 0 \right\}.$$

Show that E_0 is measurable and invariant.

Hint: To show measurability, first show that for $n \in \mathbb{N}$, $k_1, \ldots, k_n \in \mathbb{N}$ the set $\left\{ x \in X \,\middle|\, T^k(x) \in E \text{ precisely for } k = k_1, k_2, \ldots, k_n \right\}$ is measurable, then consider the complement of E_0.

Exercise 3* (Ergodicity and essential reachability)
Let $(X, \mathcal{B}, \mu, T)$ be a ppt and $A \in \mathcal{B}$ with $\mu(A) > 0$. Consider $D_A := \bigcup_{n=0} T^{-n} A$. Show that if T is ergodic, then $\mu(D_A) = 1$; that is the set of points which reach A under iteration of T is of full measure.

Hint: What can you say about $D_A \triangle T^{-1} D_A$?

Exercise 4 (Mixing: an invariant under measure-theoretic isomorphism)
Show that (strong) mixing is an invariant of measure theoretic isomorphism. That is, suppose $(X, \mathcal{B}, \mu, T)$ is a mixing ppt, and that $(X', \mathcal{B}', \mu', T')$ is a ppt which is measure theoretically isomorphic to $(X, \mathcal{B}, \mu, T)$. Show that $(X', \mathcal{B}', \mu', T')$ is mixing.

Exercise 5* (MATLAB: circle rotation)
Let $T = R_\alpha : S^1 \to S^1$, $x \mapsto x + \alpha \mod 1$, denote the rigid rotation of the unit circle by angle α. Choose $\alpha = 1/\pi$.

(a) Write a program to simulate 60 iterates of T.

(b) Plot $T^k x$ vs k.

(c) Plot $T^k x$ vs $T^{k-1} x$.

(d) Select a subinterval on the circle and test Poincaré's Recurrence Theorem by verifying that orbits continue returning into that subinterval for as long as you care to simulate.

(e) Plot $\{T^k x\}_{k=0}^{10^5}$ as points in $[0,1)$ for various initial conditions x. Based on the results, can you decide whether or not T is ergodic?

(f) Define $f, g : S^1 \to \mathbb{R}$ by $f(x) = x^2$ and $g(x) = \sin(\pi x/2)$. Numerically estimate

$$\int f \cdot g \circ T^k \, dm - \int f \, dm \cdot \int g \, dm$$

for $k = 1, \ldots, 100$, where m is the Lebesgue measure on S^1. Based on the results, can you decide whether T is mixing?