Ergodic Theory and Transfer Operators
— Question sheet 3 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Bernoulli measure)
Let $X = \{0,1\}^{\mathbb{N}}$, $\mathcal{B}$ the corresponding σ-algebra generated by the cylinders, and μ the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli measure. Show that the set

$$X' = \big\{\{x_k\} \in X \,\big|\, \nexists n \text{ s.t. } x_\ell = 1 \;\forall \ell \geq n\big\}$$

is μ-measurable and has full measure.

Hint: Show that the complement of X' is a countable set.

Remark: This result is used in the lectures (§1.15) to show the measure-theoretic isomorphy of the $(\frac{1}{2}, \frac{1}{2})$-Bernoulli scheme and the angle doubling map.

Exercise 2* (Characterization of mpts)
Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $T : X \to X$ a measurable transformation. Show that the following are equivalent:

(a) $(X, \mathcal{B}, \mu, T)$ is a mpt.

(b) For any $f \in L^\infty(X, \mu)$ it holds

$$\int_X f d\mu = \int_X f \circ T d\mu \,.$$

Hint: Simple functions are dense in L^∞.

Exercise 3 (The tent map)
Represent the tent map $T : [0,1] \to [0,1]$, $T(x) = 1 - 2|x - \frac{1}{2}|$, on $\{0,1\}^{\mathbb{N}}$.

Exercise 4* (Markov measure)
Let P be a stochastic matrix and p a stationary, componentwise positive, probability vector. Let μ denote the (p, P)-Markov measure, T the left shift, and $P_{ij}^{(n)}$ the $(i, j)^{\text{th}}$ entry of P^n. Further, let $[\underline{a}] = [a_0, \dots, a_k]$ and $[\underline{b}] = [b_0, \dots, b_\ell]$. Show that for $n > k$

$$\mu\left([\underline{a}] \cap T^{-n}[\underline{b}]\right) = p_{a_0} P_{a_0 a_1} \cdots P_{a_{k-1} a_k} P_{a_k b_0}^{(n-k)} P_{b_0 b_1} \cdots P_{b_{\ell-1} b_\ell} \,.$$