Ergodic Theory and Transfer Operators
— Question Sheet 3 —
Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Bernoulli measure)
Let \( X = \{0, 1\}^\mathbb{N} \), \( B \) the corresponding \( \sigma \)-algebra generated by the cylinders, and \( \mu \) the \((\frac{1}{2}, \frac{1}{2})\)-Bernoulli measure. Show that the set
\[
X' = \{ \{x_k\} \in X | \exists n \text{ s.t. } x_\ell = 1 \forall \ell \geq n \}
\]
is \( \mu \)-measurable and has full measure.

Hint: Show that the complement of \( X' \) is a countable set.

Remark: This result is used in the lectures (§1.15) to show the measure-theoretic isomorphy of the \((\frac{1}{2}, \frac{1}{2})\)-Bernoulli scheme and the angle doubling map.

Exercise 2* (Characterization of mpts)
Let \((X, B, \mu)\) be a finite measure space and \( T : X \to X \) a measurable transformation. Show that the following are equivalent:

(a) \((X, B, \mu, T)\) is a mpt.

(b) For any \( f \in L^\infty(X, \mu) \) it holds
\[
\int_X f d\mu = \int_X f \circ T d\mu.
\]

Hint: Simple functions are dense in \( L^\infty \).

Exercise 3 (The tent map)
Represent the tent map \( T : [0, 1] \to [0, 1], T(x) = 1 - 2|x - \frac{1}{2}|, \) on \( \{0, 1\}^\mathbb{N} \).

Exercise 4* (Markov measure)
Let \( P \) be a stochastic matrix and \( p \) a stationary, componentwise positive, probability vector. Let \( \mu \) denote the \((p, P)\)-Markov measure, \( T \) the left shift, and \( P^{(n)}_{ij} \) the \((i, j)\)th entry of \( P^n \). Further, let \( [a] = [a_0, \ldots, a_k] \) and \( [b] = [b_0, \ldots, b_\ell] \). Show that for \( n > k \)
\[
\mu ( [a] \cap T^{-n}[b] ) = p_{a_0} P_{a_0 a_1} \cdots P_{a_{k-1} a_k} P_{a_k b_0} P_{b_0 b_1} \cdots P_{b_{\ell-1} b_\ell}.
\]