ERGODIC THEORY AND TRANSFER OPERATORS
— QUESTION SHEET 4 —
Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Product)

(a) Verify Proposition 1.19 (ergodicity and mixing for products of mpts) for an example numerically. Using the circle rotation \( T_1(x) = x + a \mod 1 \) and the angle tripling map \( T_2(x) = 3x \mod 1 \), look at \( T_1 \times T_1, T_2 \times T_2, \) and \( T_1 \times T_2 \).

(b) What do you expect: is the product of an ergodic transformation and a mixing transformation always ergodic/mixing?

Exercise 2* (Skew-product)

Let \( (\Omega, \mathcal{F}, \varrho, \sigma) \) be a ppt and \( \{(X, \mathcal{B}, \mu, T_\omega)\}_{\omega \in \Omega} \) be a family of ppts. Define the skew-product \( \tau(\omega, x) = (\sigma \omega, T_\omega x) \) as in the lectures.

(a) For \( F \in \mathcal{F}, A \in \mathcal{B}, \) show \( \tau \) preserves the product measure \( \varrho \times \mu \) of sets of the form \( F \times A \); that is, show that \( (\varrho \times \mu)(\tau^{-1}(F \times A)) = (\varrho \times \mu)(F \times A) \).

Hint: Write \( (\varrho \times \mu)(\tau^{-1}(F \times A)) \) as an integral w.r.t. \( \omega \) using Fubini’s theorem.

Remark: This is the first step in showing that \( \tau \) preserves the product measure \( \varrho \times \mu \) of all measurable sets in \( \mathcal{F} \otimes \mathcal{B} \).

(b) Show that the skew-product is an extension of its base \( (\Omega, \mathcal{F}, \varrho, \sigma) \); that is, the base is a factor of the skew-product.

Exercise 3* (A mixing factor)

Give an example of two mpts \( \text{mpt}_1 : (X, \mathcal{B}, \mu, T) \) and \( \text{mpt}_2 : (X, \mathcal{B}', \mu', T') \), such that \( \text{mpt}_2 \) is a factor of \( \text{mpt}_1, \mathcal{B}' \subset \mathcal{B} \), and \( \text{mpt}_2 \) is mixing, while \( \text{mpt}_1 \) is not.