ERGODIC THEORY AND TRANSFER OPERATORS
— QUESTION SHEET 4 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Product)

(a) Verify Proposition 1.19 (ergodicity and mixing for products of mpts) for an example numerically. Using the circle rotation $T_1(x) = x + \alpha \mod 1$, and the angle tripling map $T_2(x) = 3x \mod 1$, look at $T_1 \times T_1$, $T_2 \times T_2$, and $T_1 \times T_2$.

(b) What do you expect: is the product of an ergodic transformation and a mixing transformation always ergodic/mixing?

Exercise 2* (Skew-product)
Let $(\Omega, \mathcal{F}, \varrho, \sigma)$ be a ppt and $\{(X, \mathcal{B}, \mu, T_\omega)\}_{\omega \in \Omega}$ be a family of ppts. Define the skew-product $\tau(\omega, x) = (\sigma\omega, T_\omega x)$ as in the lectures.

(a) For $F \in \mathcal{F}$, $A \in \mathcal{B}$, show τ preserves the product measure $\varrho \times \mu$ of sets of the form $F \times A$; that is, show that $(\varrho \times \mu)(\tau^{-1}(F \times A)) = (\varrho \times \mu)(F \times A)$.

Hint: Write $(\varrho \times \mu)(\tau^{-1}(F \times A))$ as an integral w.r.t. ω using Fubini's theorem.

Remark: This is the first step in showing that τ preserves the product measure $\varrho \times \mu$ of all measurable sets in $F \otimes \mathcal{B}$.

(b) Show that the skew-product is an extension of its base $(\Omega, \mathcal{F}, \varrho, \sigma)$; that is, the base is a factor of the skew-product.

Exercise 3* (A mixing factor)
Give an example of two mpts $mpt_1 : (X, \mathcal{B}, \mu, T)$ and $mpt_2 : (X, \mathcal{B}', \mu', T')$, such that mpt_2 is a factor of mpt_1, $\mathcal{B}' \subset \mathcal{B}$, and mpt_2 is mixing, while mpt_1 is not.