Ergodic Theory and Transfer Operators
— Question Sheet 6 —
Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Reachability Revisited)
Let \( X \subset \mathbb{R}^d \) with \( \mu(X) < \infty \), where \( \mu \) is the Lebesgue measure. Let \( T : X \to X \) be ergodic with respect to \( \mu \). Show by using the Birkhoff Ergodic Theorem that for almost all \( x \in X \) the sequence \( \{T^kx\}_{k \in \mathbb{N}} \) is dense in \( X \).

Remark: Note that this is a slightly stronger version of Exercise 2.3.

Exercise 2 (Mixing)
Let \((X, \mathcal{B}, \mu, T)\) be a mixing ppt. Let \( \nu \) be a probability measure on \( \mathcal{B} \) absolutely continuous with respect to \( \mu \). Show that \( \lim_{n \to \infty} \nu(T^{-n}A) = \mu(A) \) for \( A \in \mathcal{B} \).

Hint: Express \( \nu(T^{-n}A) \) by the Radon–Nikodým derivative \( \frac{d\nu}{d\mu} \), and use Proposition 1.13B from the lectures.

Remark: Stochastic interpretation. What does this result tell you about the distribution of \( T^n x \), if \( x \) is a random variable with distribution \( \nu \)?

Exercise 3* (Correlation)
Let \( P \in \mathbb{R}^{s \times s} \) be an irreducible stochastic matrix with stationary vector \( p \). Let \( X_1, X_2, \ldots \) be a sequence of random variables with \( X_1 \sim p \) (i.e., \( X_1 \) is distributed according to \( p \)) and
\[
\text{Prob}(X_{k+1} = j \mid X_k = i) = P_{ij},
\]
i.e. it is a Markov chain. Let \( f, g : \{1, \ldots, s\} \to \mathbb{R} \). Does the sequence
\[
\frac{1}{n} \sum_{k=1}^{n} f(X_k)g(X_{k+1}), \quad n = 1, 2, \ldots
\]
converge almost surely as \( n \to \infty \)? If yes, determine the limit.

Hint: The problem looks similar to Exercise 5.3...