ERGODIC THEORY AND TRANSFER OPERATORS
— QUESTION SHEET 6 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Reachability Revisited)
Let $X \subset \mathbb{R}^d$ with $\mu(X) < \infty$, where μ is the Lebesgue measure. Let $T : X \to X$ be ergodic with respect to μ. Show by using the Birkoff Ergodic Theorem that for almost all $x \in X$ the sequence $\{T^k x\}_{k\in\mathbb{N}}$ is dense in X.

Remark: Note that this is a slightly stronger version of Exercise 2.3.

Exercise 2 (Mixing)
Let $(X, \mathcal{B}, \mu, T)$ be a mixing ppt. Let ν be a probability measure on $\mathcal{B}$ absolutely continuous with respect to μ. Show that $\lim_{n\to\infty} \nu(T^{-n}A) = \mu(A)$ for $A \in \mathcal{B}$.

Hint: Express $\nu(T^{-n}A)$ by the Radon–Nikodým derivative $\frac{d\nu}{d\mu}$, and use Proposition 1.13B from the lectures.

Remark: Stochastic interpretation. What does this result tell you about the distribution of $T^n x$, if x is a random variable with distribution ν?

Exercise 3* (Correlation)
Let $P \in \mathbb{R}^{s\times s}$ be an irreducible stochastic matrix with stationary vector p. Let $X_1, X_2, \dots$ be a sequence of random variables with $X_1 \sim p$ (i.e., X_1 is distributed according to p) and

$$\mathrm{Prob}(X_{k+1} = j \mid X_k = i) = P_{ij}\,,$$

i.e. it is a Markov chain. Let $f, g : \{1, \dots, s\} \to \mathbb{R}$. Does the sequence

$$\frac{1}{n}\sum_{k=1}^{n} f(X_k)g(X_{k+1}), \qquad n = 1, 2, \dots$$

converge almost surely an $n \to \infty$? If yes, determine the limit.

Hint: The problem looks similar to Exercise 5.3...