Ergodic Theory and Transfer Operators
— Question sheet 7 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1 (Conditional expectation)
Let $(X, \mathcal{B}, \mu)$ be a measure space and $\mathcal{A} \subset \mathcal{B}$ another σ-algebra. Show that

(a) $f \mapsto \mathbb{E}(f|\mathcal{A})$ is linear and a contraction (non-expansion) in the L^1-norm;

(b) if $f \geq 0$, then $\mathbb{E}(f|\mathcal{A}) \geq 0$;

(c) if $\phi : \mathbb{R} \to \mathbb{R}$ is convex, then $\mathbb{E}(\phi \circ f|\mathcal{A}) \geq \phi\big(\mathbb{E}(f|\mathcal{A})\big)$;

Hint: A convex function is the supremum of countably many affine-linear functions.

(d) if g is $\mathcal{A}$-measurable and essentially bounded, then $\mathbb{E}(gf|\mathcal{A}) = g\,\mathbb{E}(f|\mathcal{A})$;

(e) if $\mathcal{F} \subset \mathcal{A}$ is a σ-algebra, then $\mathbb{E}\big(\mathbb{E}(f|\mathcal{A})|\mathcal{F}\big) = \mathbb{E}(f|\mathcal{F})$;

(f) $\operatorname{esssup} f \geq \operatorname{esssup} \mathbb{E}(f|\mathcal{A})$; and

Hint: For an $\mathcal{A}$-measurable function f holds $\operatorname{esssup} f > M$ for some $M \in \mathbb{R} \Leftrightarrow$ there exists an $A \in \mathcal{A}$ with $\mu(A) > 0$ such that $f > M$ a.e. on A.

(g) if $X = [-1, 1]$, $\mathcal{B}$ the Borel σ-algebra, μ the Lebesgue measure, and $\mathcal{A} = \{A \in \mathcal{B} \,|\, A = -A\}$, then $\mathbb{E}(f|\mathcal{A})(x) = \frac{1}{2}(f(x) + f(-x))$.

Exercise 2* (Transfer operators and the support)
Let $T : X \to X$ a non-singular transformation, P and U the associated Frobenius–Perron, and Koopman operators, respectively. Further, let $f \in L^1(X)$ and $g \in L^\infty(X)$. Show that, up to sets of measure zero,

(a) $\operatorname{supp} Ug = T^{-1}(\operatorname{supp} g)$; and

(b) $\operatorname{supp} Pf \subseteq T(\operatorname{supp} f)$,

where $\operatorname{supp} h$ denotes the support of the function h, i.e. the set $\{h \neq 0\}$ (which is thus defined up to a null set).

Hint: For (b), assume the counterpart, i.e. that $M := \operatorname{supp} Pf \setminus T(\operatorname{supp} f)$ has positive Lebesgue measure, and consider Pf on subsets of M.

(c) Give an example where the inclusion in (b) is strict.

Exercise 3* (Weak mixing)
We call the ppt $(X, \mathcal{B}, \mu, T)$ *weakly mixing* if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \mu\big(T^{-k}A \cap B\big) - \mu(A)\mu(B) \right| = 0 \qquad \forall\, A, B \in \mathcal{B}\,.$$

We have the following result [Wa, Theorem 1.24].

Theorem 1: The following are equivalent:

(i) $(X, \mathcal{B}, \mu, T)$ is weakly mixing.

(ii) $(X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu, T \times T)$ is ergodic.

Show the following:

(a) If T is mixing, then it is weakly mixing.

(b) If T is weakly mixing, then it is ergodic.

(c) Let $X \subset \mathbb{R}^d$ be open, and $\mathcal{B}$ the Borel σ-algebra induced by X. If T is weakly mixing, then for $\mu \times \mu$-almost pair (x, y) it holds

$$\liminf_{n \to \infty} \|T^n x - T^n y\| = 0\,.$$

Hint: Use the theorem above and Exercise 2.3.