Ergodic Theory and Transfer Operators
— Question sheet 8 —

Summer 2017

Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

Exercise 1* (Monte Carlo estimation of the Ulam matrix)

Let $\hat{P}_n$ be the numerical approximation by Monte Carlo sampling of the Ulam matrix P_n from the lectures, §3.10. Show that $\hat{P}_n$, as a linear mapping $c \mapsto \hat{P}_n c$, is a collocation-type Pertov–Galerkin projection of the Koopman operator U.

Exercise 2* (Ulam's method for mixing transformations)

Let $(X, \mathcal{B}, \mu, T)$ be a probability preserving transformation. For some partition $\mathscr{P}_n = \{B_1, \ldots, B_n\}$, let $P_n \in \mathbb{R}^{n\times n}$ denote the Ulam matrix, i.e.

$$P_{n,ij} = \frac{\mu\left(B_i \cap T^{-1}B_j\right)}{\mu(B_i)}.$$

(a) Show that $p_n \in \mathbb{R}^n$ with $p_{n,i} = \mu(B_i)$ is a stationary vector of P_n.

(b) Show that if T is mixing, so is the (p_n, P_n)-Markov shift.

Exercise 3 (Change of measure)

Let μ and ν be equivalent probability measures; that is $\mu \ll \nu$ and $\nu \ll \mu$. Show, that if $h \in \mathcal{D}(X, \mathcal{B}, \nu)$ is the Radon–Nikodým derivative of μ w.r.t. ν, then for any $f \in L^1(X, \mathcal{B}, \mu)$

$$P_{T,\mu} f = \frac{P_{T,\nu}(fh)}{h},$$

where $P_{T,\mu}$ and $P_{T,\nu}$ denote the Frobenius–Perron operator associated with the nonsingular transformation $T : X \to X$ w.r.t. the measures μ and ν, respectively.