Please solve and prepare one of the exercises marked by * for the exercise session on the week after the question sheet appears.

**Exercise 1** (Monte Carlo estimation of the Ulam matrix)

Let \( \hat{P}_n \) be the numerical approximation by Monte Carlo sampling of the Ulam matrix \( P_n \) from the lectures, §3.10. Show that \( \hat{P}_n \), as a linear mapping \( c \mapsto \hat{P}_n c \), is a collocation-type Pertov–Galerkin projection of the Koopman operator \( \mathcal{U} \).

**Exercise 2** (Ulam’s method for mixing transformations)

Let \((X, \mathcal{B}, \mu, T)\) be a probability preserving transformation. For some partition \( \mathcal{P}_n = \{B_1, \ldots, B_n\} \), let \( P_n \in \mathbb{R}^{n \times n} \) denote the Ulam matrix, i.e.

\[
P_{n,ij} = \frac{\mu(B_i \cap T^{-1}B_j)}{\mu(B_i)}.
\]

(a) Show that \( p_n \in \mathbb{R}^n \) with \( p_{n,i} = \mu(B_i) \) is a stationary vector of \( P_n \).

(b) Show that if \( T \) is mixing, so is the \((p_n, P_n)\)-Markov shift.

**Exercise 3** (Change of measure)

Let \( \mu \) and \( \nu \) be equivalent probability measures; that is \( \mu \ll \nu \) and \( \nu \ll \mu \). Show, that if \( h \in D(X, \mathcal{B}, \nu) \) is the Radon–Nikodým derivative of \( \mu \) w.r.t. \( \nu \), then for any \( f \in L^1(X, \mathcal{B}, \mu) \)

\[
P_{T,\mu} f = \frac{P_{T,\nu}(fh)}{h},
\]

where \( P_{T,\mu} \) and \( P_{T,\nu} \) denote the Frobenius–Perron operator associated with the nonsingular transformation \( T : X \to X \) w.r.t. the measures \( \mu \) and \( \nu \), respectively.