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Exercise 2 for the lecture

Numerics III

SoSe 2018

http://numerik.mi.fu-berlin.de/wiki/SS_2018/NumericsIII.php

Due: Wed, 05-09-2018

Problem 1 (4 TP)

Let V be normed space over $\mathbb{R}$ with norm $\|\cdot\|$. Prove that $\|\cdot\|$ is induced by some inner product $\langle\cdot,\cdot\rangle$ i.e. $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ if and only if the parallelogram identity

$$2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2 \tag{1}$$

holds for all $u, v \in V$.

Give an interpretation of the parallelogram identity (1).

Problem 2 (4 TP)

Let $\Omega = (-1,1)$ and $\|u\|_{H^1(\Omega)} := \sqrt{\|u\|^2_{L^2(\Omega)} + \|u'\|^2_{L^2(\Omega)}}$ for all $u \in C^1(\Omega)$. Show that

a) the space $C^0(\overline{\Omega})$ is not complete with respect to $\|\cdot\|_{L^2(\Omega)}$ and

b) the space $C^1(\overline{\Omega})$ is not complete with respect to $\|\cdot\|_{H^1(\Omega)}$.

Problem 3 (5 TP)

The (nonlinear) Cauchy-Green strain tensor and the linear strain tensor for a smooth displacement vector field $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ are given by

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{l=1}^{d} \frac{\partial u_l}{\partial x_i}\frac{\partial u_l}{\partial x_j}\right), \qquad \mathcal{E}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Explain why the resulting displacement of a rigid body motion applied after u is given by $\tilde{u}(x) = A(x + u(x)) - x + c$ for an orthogonal matrix A and a vector c. Show that

E is invariant under any rigid body motion while $\mathcal{E}$ is only invariant under translations but not under rotations.

Problem 4 (4 TP)
Let $\langle \cdot, \cdot \rangle_{M_m}$ and $\langle \cdot, \cdot \rangle_{M_n}$ be inner products on $\mathbb{R}^m$ and $\mathbb{R}^n$ induces by s.p.d. matrices $M_m \in \mathbb{R}^{m\times m}$ and $M_n \in \mathbb{R}^{n\times n}$, respectively. Furthermore let $D \in \mathbb{R}^{m\times n}$ and $D^* \in \mathbb{R}^{n\times m}$ the adjoint matrix in the sense that $\langle Dx, y\rangle_{M_m} = \langle x, D^* y\rangle_{M_n}$ for all $x \in \mathbb{R}^n, y \in \mathbb{R}^m$. For $f \in \mathbb{R}^n$ consider the energy functional

$$J(v) = \frac{1}{2}\langle Dv, Dv\rangle_{M_m} - \langle f, v\rangle_{M_n}.$$

a) Show that the minimization problem

$$u \in \mathbb{R}^n : \qquad J(u) \leq J(v) \qquad \forall v \in \mathbb{R}^n$$

is equivalent to the variational equation

$$u \in \mathbb{R}^n : \qquad \langle Du, Dv\rangle_{M_m} = \langle f, v\rangle_{M_n} \qquad \forall v \in \mathbb{R}^n.$$

b) Show that the variational equation is equivalent to the equation

$$D^* Du = f,$$

give the matrix $D^* D$, and show that it is symmetric, positive semi-definite.

c) Discuss, in which sense the results are analogues to the results for the membrane deflection problem by relating $\langle \cdot, \cdot \rangle_{M_m}$, $\langle \cdot, \cdot \rangle_{M_n}$, D, D^*, and any of the above problem formulations to their contiuous counterpart.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.