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Exercise 3 for the lecture

Numerics III

SoSe 2018

http://numerik.mi.fu-berlin.de/wiki/SS_2018/NumericsIII.php

Due: Wed, 05-16-2018

Problem 1 (8 TP)

a) Suppose a map $A \in C(\mathbb{R}^n, \mathbb{R}^{n\times n})$, a vector field $b \in C(\mathbb{R}^n, \mathbb{R}^n)$ and a scalar function $c \in C(\mathbb{R}^n, \mathbb{R})$. The differential operator defined by

$$L(x, u) := A(x) : D^2u(x) + b(x) \cdot \nabla u(x) + c(x)u(x)$$

is called *elliptic in* $x \in \mathbb{R}^n$ if and only if all eigenvalues of $A(x)$ are unequal 0 and have the same sign.

Let $n = 2$. Show that L is elliptic in $x \in \mathbb{R}^n$ in the sense of (a) if and only if $A_{11}(x)A_{22}(x) - A_{12}(x)A_{21}(x) > 0$.

b) Let $m \in \mathbb{N}$ and $a_\alpha \in C(\mathbb{R}^n, \mathbb{R})$ for every multi-index α with $|\alpha| \leq m$. The differential operator defined by

$$L(x, u) := \sum_{|\alpha|\leq m} a_\alpha(x)\partial_\alpha u(x)$$

is called *elliptic in* $x \in \mathbb{R}^n$ if and only if for all $\xi \in \mathbb{R}^n \setminus \{0\}$

$$L'(x, \xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0.$$

Show that every elliptic differential operator in the sense of (a) is elliptic in the sense of (b).

c) Suppose that $n \geq 2$ and that L is an elliptic differential operator in the sense of (b). Prove that m is even.

Remarks

- For two matrices $A, B \in \mathbb{R}^{n\times n}$ we define $A : B := \sum_{i,j=1}^{n} A_{ij}B_{ij}$ to be the sum of the element-wise products of A and B.
- An operator in the sense of (a) is called *parabolic* if A has one zero-eigenvalue and all other eigenvalues are unequal 0 and have the same sign. The operator L is called *hyperbolic* if A has one strictly negative (or positive) eigenvalue and all other eigenvalues are strictly positive (or negative).
- For a vector $\xi \in \mathbb{R}^n$ and a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ we define $\xi^\alpha := \prod_{i=1}^{n} \xi_i^{\alpha_i}$.
- The last statement is even true for complex coefficients if $n \geq 3$.

Problem 2 (4 TP)

a) Define the transformation map from polar coordinates to Cartesian coordinates by

$$X\colon \mathbb{R}_0^+ \times \mathbb{R} \longrightarrow \mathbb{R}^2, \qquad (r,\varphi) \longmapsto r \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}.$$

For $u \in C^2(\mathbb{R}^2 \setminus \{0\})$ and $\widehat{u} = u \circ X \in C^2(\mathbb{R}^+ \times \mathbb{R})$ show that

$$(\Delta u)(X(r,\varphi)) = \partial_{rr}\widehat{u}(r,\varphi) + \frac{1}{r}\partial_r \widehat{u}(r,\varphi) + \frac{1}{r^2}\partial_{\varphi\varphi}\widehat{u}(r,\varphi) \qquad \forall (r,\varphi) \in \mathbb{R}^+ \times \mathbb{R}.$$

b) Let $u(x) = \log|x|$. Show that u is harmonic on $\mathbb{R}^2 \setminus \{0\}$.

Problem 3 (4 TP)
Let Ω be bounded and open set in $\mathbb{R}^n$.
a) If $u \in C^2(\Omega) \cup C(\overline{\Omega})$ is a harmonic function, then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \qquad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

b) For a given functions $f \in C(\Omega)$ and $g \in C(\partial\Omega)$ there exists at most one solution $u \in C^2(\Omega) \cup C(\overline{\Omega})$ of the Poisson problem with Dirichlet boundary conditions.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.