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Exercise 4 for the lecture

Numerics III

SoSe 2018

http://numerik.mi.fu-berlin.de/wiki/SS_2018/NumericsIII.php

Due: Fri, 05-25-2018

Problem 1 (6 TP)

a) Let $u \in C^2(\Omega)$ be a harmonic function in an open domain Ω. Show that

$$u(x) = \fint_{B_r(x)} u(x)dx \qquad u(x) = \fint_{\partial B_r(x)} u(x)dS. \tag{1}$$

b) Let $u \in C(\Omega)$ satisfies (1). Prove that then $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

c) Let $u \in C^2(\Omega)$ be harmonic and $B_r(x) \subset \Omega$. Show that

$$|\partial_i u(x)| \leq \frac{n}{r} \max_{\overline{B}_r(x)} |u|.$$

d) Prove that if $u \in C^2(\mathbb{R}^n)$ is bounded and harmonic in $\mathbb{R}^n$ then u is constant.

Problem 2 (6 TP)

Let $\Omega \subseteq \mathbb{R}^n$ be bounded with smooth boundary. Furthermore, suppose that $\gamma \in C^1(\mathbb{R}^n, \mathbb{R})$ is strictly convex and define $J(u) := \int_\Omega \gamma(\nabla u(x))\,\mathrm{d}x$ for all $u \in C^1(\overline{\Omega})$.

a) For each $f \in C^1(\overline{\Omega})$ define $V_f := \{u \in C^1(\overline{\Omega}) \mid u|_{\partial\Omega} = f|_{\partial\Omega}\}$. Show that J is strictly convex on the set V_f for any $f \in C^1(\overline{\Omega})$.

b) Show that there can be at most one $u^* \in V_f$ such that $J(u^*) = \inf_{u \in V_f} J(u)$.

c) From now on assume that the minimizer $u^* \in V_f$ of J over V_f exists. Show that

$$\forall v \in V_0\colon \int_\Omega \nabla\gamma(\nabla u^*(x)) \cdot \nabla v(x)\,\mathrm{d}x = 0.$$

d) Assume furthermore that $\gamma(x) = \sqrt{1 + \|x\|^2}$ for all $x \in \mathbb{R}^n$ and that $u^* \in C^2(\overline{\Omega})$. Prove that for every $x \in \Omega$

$$\operatorname{div}\left(\frac{\nabla u^*(x)}{\sqrt{1 + \|\nabla u^*(x)\|^2}}\right) = 0.$$

Problem 3 (4 TP)
Two norms $\|\cdot\|_A$ and $\|\cdot\|_B$ on a normed vector space V are equivalent if there are constants $c_1, c_2 > 0$ with

$$c_1\|v\|_A \leq \|v\|_B \leq c_2\|v\|_A \qquad \forall v \in V.$$

Show that $\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}$ is equivalent to the norm $\|v\|_1$ on $H^1(\Omega)$ given by

$$\|v\|_1 = \sqrt{(v,v)_1}, \qquad (u,v)_1 = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}$$

with constants independent on the dimension and Ω.

Problem 4 (8 PP)
Let $\Omega := [0,1]^2$ and $f, g \in C(\overline{\Omega})$.

a) For each $N \in \mathbb{N}$ let $h := \frac{1}{N}$ and $\Omega_N := \{(ih, jh) \mid (i,j) \in \{0, \dots, N\}^2\}$. Implement the Shortley-Weller method for the problem

$$\begin{aligned} -\Delta_h U(x) &= f(x) \text{ for each } x \in \Omega_N \\ U(x) &= g(x) \text{ for each } x \in \partial\Omega_N \end{aligned} \tag{2}$$

by writing functions

`A` = OperatorAssembler(N)
`F` = FunctionalAssembler(f, g, N)

which assemble the coefficient matrix A and the right-hand side vector F of this discrete equation for given function handles `f` and `g`. The returned matrix should be stored in a sparse format.

b) Let

$$\begin{aligned} f_1 &= 2(-x_1^2 - x_2^2 + x_1 + x_2) & g_1 &= 0 & u_1 &= (x_1^2 - x_1)(x_2^2 - x_2) \\ f_2 &= -4 & g_2 &= \|x\|^2 & u_2 &= \|x\|^2 \\ f_3 &= -2\pi^2 \sin(\pi x_1)\sin(\pi x_2) & g_3 &= 0 & u_3 &= \sin(\pi x_1)\sin(\pi x_2). \end{aligned}$$

Solve the discrete problem (2) given (f_k, g_k) and $N = 2^l$ for all $k \in \{1,2,3\}$ and $l \in \{2, \dots, 8\}$. Plot the graphs of the discrete solutions U. In addition, plot the errors $\max_{x \in \Omega_N} |u_k(x) - U_k(x)|$ versus $h = \frac{1}{N}$ in a logarithmic scale with 1:1 aspect-ratio. You can use the command `axis equal` and the function `loglog`.

Remark It is necessary that you add comments to your code which explain your implementation.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsIII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.