

Exercise 5 for the lecture

NUMERICS III

SoSe 2018

http://numerik.mi.fu-berlin.de/wiki/SS_2018/NumericsIII.php

Due: Wed, 06-06-2018

Problem 1 (4 TP)

Let H be a Hilbert space, $a(\cdot, \cdot)$ be a symmetric, H -elliptic bilinear form with $\|v\|_a := \sqrt{a(v, v)}$, $l \in H'$ and $S \subset H$ be a closed subspace. Furthermore, let u and u_s be solutions of the following variational problems

$$u \in H : \quad a(u, v) = l(v) \quad \forall v \in H$$

$$u_s \in S : \quad a(u_s, v) = l(v) \quad \forall v \in S.$$

Show that

$$\|u - u_s\|_a = \inf_{w \in S} \|u - w\|_a \quad \text{and} \quad \|u - u_s\|_H \leq \sqrt{\frac{\Gamma}{\gamma}} \inf_{w \in S} \|u - w\|_H.$$

Problem 2 (6 TP)

Let $\Omega \subset \mathbb{R}$ be open, bounded set.

a) Let $f \in L^2(\Omega)$. Show that

$$\tilde{l}(v) := \int_{\Omega} f(x)v(x)dx$$

defines a linear continuous functional on $(L^2(\Omega), \|\cdot\|_2)$ and $(C_0^1(\overline{\Omega}), \|\cdot\|_{1,\infty})$.

b) Let $\Omega = (-1, 1)$

$$g(x) := \begin{cases} 0 & \text{if } -1 < x < 0 \\ 1 & \text{if } 0 \leq x < 1 \end{cases}$$

and

$$l(v) := - \int_{-1}^1 g(x)v'(x)dx.$$

Show that l is a linear continuous functional on $(C_0^1(\bar{\Omega}), \|\cdot\|_{1,\infty})$.

- c) Show that $l(v) = v(0)$.
- d) Show that there is no function $f \in L^2(\Omega)$ such that $l(v) = \int_{\Omega} f(x)v(x)dx$.

Problem 3 (3 TP)

Let $\Omega = (a, b)$ and $f \in C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) : \text{supp } f \subset \Omega\}$. Show that all derivatives of f vanish in a neighborhood of the boundary $\partial\Omega$.

Problem 4 (3 TP + 3 TP extra)

- a) Let Ω be bounded. Prove that for $1 \leq q \leq p \leq \infty$, it holds $L^p(\Omega) \subset L^q(\Omega)$.
- b) (extra points) Give an example that $L^2(\Omega) \subset L^1(\Omega)$ doesn't hold for an unbounded Ω .

Remark: You can find the definition of $L^p(\Omega)$ for example in Alt [1].

GENERAL REMARKS

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to adjurdjevac@mi.fu-berlin.de with a subject starting by [NumericsIII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.