Fachbereich Mathematik & Informatik
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Exercise 6 for the lecture

Numerics III

SoSe 2018

http://numerik.mi.fu-berlin.de/wiki/SS_2018/NumericsIII.php

Due: Wed, 06-13-2018

Problem 1 (3 TP)
Let $\Omega = \mathbb{R}$. Determine the weak derivative of the function $u(x) = (1 - |x|)_+, x \in \Omega$.

Problem 2 (6 TP)
Let $u \in C^1(a,b)$. Prove the following inequalities.

a)
$$\int_a^b u^2(x)dx \leq 2(b-a)^2\|u'\|_0^2 + 2\frac{1}{(b-a)}\left(\int_a^b u(x)dx\right)^2$$
.

b)
$$\int_a^b (u(x) - A)^2 \, dx \leq (b-a)^2\|u'\|_0^2,$$
where $A := ⨍_a^b u(x)dx = \frac{1}{b-a}\int_a^b u(x)dx$.

c)
$$\|u\|_0^2 \leq (b-a)^2\|u'\|_0^2 + \frac{1}{b-a}\left(\int_a^b u(x)dx\right)^2$$
.

Problem 3 (2 TP)
Let $\Phi : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ maps any vector $v \in \mathbb{R}^d$ to orthogonal basis given by the column vectors of $\Phi(x)$ for any $x \in \mathbb{R}^d$ i.e. $\Phi(x)$ is orthogonal matrix for any x. Show that the Laplace operator in the weak form is invariant under local rotations. More precisely, show that
$$\int_\Omega \nabla u(x) \cdot \nabla v(x)dx = \int_\Omega \nabla_{\Phi(x)}u(x) \cdot \nabla_{\Phi(x)}v(x)dx, \quad \forall u, v \in H^1(\Omega),$$

where $\Omega \subset \mathbb{R}$ is open and bounded and $\nabla_{\Phi(x)}$ is the gradient computed in the coordinate of $\Phi(x)$ basis.

Problem 4 (4 TP)
Let $\Omega = B_1(0) \subseteq \mathbb{R}^2$. Show that the map u defined by $u(x) = \log\log\left(1 + \|x\|^{-1}\right)$ belongs to $H^1(\Omega)$.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to `adjurdjevac@mi.fu-berlin.de` with a subject starting by [NumericsIII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.