$u \in H^2(\Omega)$.

$$\| u - u_h \|_1 \leq \inf_{v \in S} \| u - v \|_1$$

$$\leq \| u - I_h u \|_1$$

$$I_h u = \sum_{p \in N_h} u(p) \lambda_p \quad m = 1$$

1. localization

$$\| u - I_h u \|^2_{1,\Omega} = \sum_{t \in T_h} \| u - I_h u \|^2_{1,t}$$

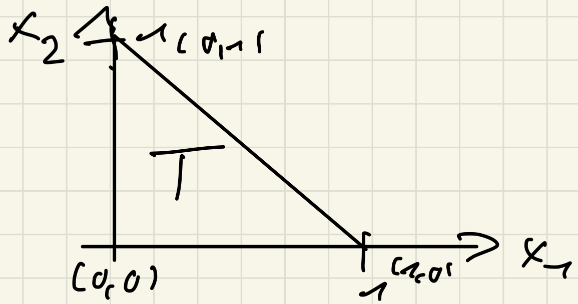
$$\| v \|^2_{1,t} = \int_t |v(x)|^2 dx + \sum_{|\beta|=1} \int_t |\partial^\beta u|^2 dx$$

2. transformation to reference element

$d=1$: $T = [0,1]$

$d=2$: $T = \{ (x_1,x_2)^T \in \mathbb{R}^2 \mid$

$x_1, x_2 \geq 0,\ x_1 + x_2 \leq 1 \}$



$d=3$: $T = \{ x \in \mathbb{R}^3 \mid x_i \geq 0,\ \sum_{i=1}^{3} x_i \leq 1 \}$

$t \in \mathcal{T}_h$ arbitrary

$F_t(T) = t$ affine linear

$$t \ni x = F_t(\xi) = x_0 + \underline{B}\,\xi, \quad \xi \in T$$

$$\hat{v}(\xi) = v(F_t(\xi))$$

$t \in \mathcal{T}_h$ arbitrary

$F_t(T) = t$ affine linear

$t \ni x = F_t(\xi) = x_0 + B\xi, \quad \xi \in T$

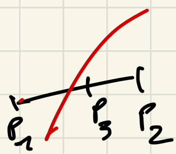
t has vertices p_1, p_2, p_3

$x_0 = p_1 \quad T \ni (0,0) \leftrightarrow p_1$

$B = (p_2, p_3) \quad T \ni (1,0) \leftrightarrow p_2$

$T \ni (0,1) \leftrightarrow p_3$

$F_t^{-1}(x) = B^{-1}(x - x_0)$

B regular, because  not 

transformation of norms

$$\|v\|_{1,t}^2 = \underline{\|v\|_{0,t}^2} + \|v_{x_1}\|_{0,t}^2 + \|v_{x_2}\|_{0,t}^2$$

$$v : t \to \mathbb{R}$$

$$\hat{v}(\xi) := v(F_t(\xi)) \quad \xi \in T$$

$$\|\hat{v}\|_{1,T} = ?$$

transformation of norms

$$\|v\|_{1,t}^2 = \underline{\|v\|_{0,t}^2} + \|v_{x_1}\|_{0,t}^2 + \|v_{x_2}\|_{0,t}^2$$

chain rule $v_{x_1}(x) = \frac{\partial}{\partial x_1} \hat{v}(F_t^{-1}(x))$

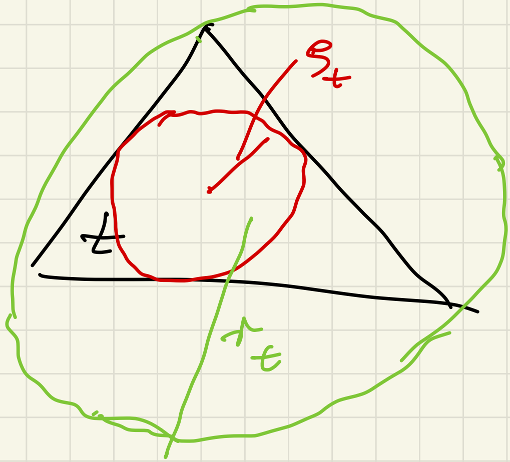
$$= \nabla_\xi \hat{v}(F_t^{-1}(x))(B^{-1})_1$$

$$\hat{v}(\xi) = v(F(\xi))$$

$$\|v\|_{1,t} \leq \|B^{-1}\| \, |\det B|^{1/2} \, \|\hat{v}\|_{1,T}$$

$$\|u - I_h u\|_{1,t} \leq \|B^{-1}\| \, |\det B|^{1/2} \, \widehat{\|u - I_h u\|}_{1,T}$$

$$= \|B^{-1}\| \, |\det B|^{1/2} \, \|\hat{u} - I \hat{u}\|_{1,T}$$



$r_t = \mathrm{diam}(t)$
$= \frac{1}{2} h_t$

$h = \max_{t \in \mathcal{T}_h} h_t$

$$\|B\| \leq \frac{r_t}{\hat{\varrho}} \qquad \hat{\varrho} = (2+\sqrt{2})^{-1}$$

$$\|B^{-1}\| \leq \frac{\hat{r}}{\varrho_t} \qquad \hat{r} = 2^{-1/2}$$

3. interpolation error on T

$d = 1$: see manuscript.

$d \in \mathbb{N}$: Bramble-Hilbert lemma

$$\| \hat{u} - I \hat{u} \|_{1,T} \le C_T \, | \hat{u} |_{2,T}$$

$$| v |_{2,T}^2 = \sum_{|\beta| = 2} \| \partial^\beta u \|_{0,T}^2$$

4. back transformation

$$| \hat{u} |_{2,T} \le \| B \|^2 \, | \det B |^{-1/2} \, | u |_{2,t}$$

local error estimate:

$$\| u - I_h u \|_{1,t} \overset{2.}{\leq} \|B^{-1}\| \, |\det B|^{1/2} \, \|\hat{u} - I \hat{u}\|_{1,T}$$

$$\overset{3.}{\leq} C_T \|B^{-1}\| \, |\det B|^{1/2} \, |\hat{u}|_{2,T}$$

$$\leq C_T \|B^{-1}\| \, \cancel{|\det B|^{1/2}} \, \underbrace{\|B\|^2}_{\lesssim r_t h_t} \, \cancel{|\det B|^{-1/2}} \, |u|_{2,t}$$

$$\leq \tilde{C}_T \, \frac{r_t}{\varrho_t} \, h_t \, |u|_{2,t}$$

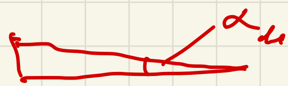
5. Final error estimate

$$\|u - I_h u\|_1^2 = \sum_{t \in \mathcal{T}_h} \|u - I_h\|_{1,t}^2$$

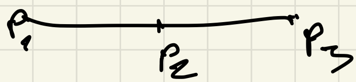
$$\|u - I_h u\|_1 \le c_T \underbrace{\max_{t \in \mathcal{T}_h} \frac{r_t}{\rho_t}}_{=:\sigma_h} h\, |u|_2$$

shape regularity of $\mathcal{T}_h$: σ_h

$\sigma_h \to \infty$:



interior angle $\alpha_h \to \infty$



discretization error estimate

$$\|u - u_h\| \le c\, \sigma_h\, h\, |u|_2$$

quasi-optimal

can we prove

$$\|u-u_h\|_0 \leq c\,\sigma_h\, h^2 \|u\|_2$$

?

Céa's lemma:

$$\|u-u_h\|_1 \leq c \inf_{v \in H} \|u-v\|_1$$

we do not have

$$\|u-u_h\|_0 \not\leq c \inf_{v \in H} \|u-v\|_0$$

Aubin - Nitsche - Trick