The Dirichlet Problem[1]

Lars Gårding

Dirichlet's problem is one of the fundamental boundary problems of physics. It appears in electrostatics, heat conduction, and elasticity theory and it can be solved in many ways. For the mathematicians of the nineteenth century it was a fruitful challenge that they met with new methods and sharper tools. I am going to give a sketch of the problem and its history.

Dirichlet did most of his work in number theory and is best known for having proved that every arithmetic progression $a, a + b, a + 2b, \ldots$ contains infinitely many primes when a and b are relatively prime. In 1855, towards the end of his life, he moved from Berlin to Göttingen to become the successor of Gauss. In Berlin he had lectured on many things including the grand subjects of contemporary physics, electricity, and heat conduction. Through one of his listeners, Bernhard Riemann, Dirichlet's name became attached to a fundamental physical problem. In bare mathematical terms it can be stated as follows.[2]

A real-valued function $u(x) = u(x_1, \ldots, x_n)$ from an open part Ω of $\mathbb{R}^n$ is said to be *harmonic* there if $\Delta u = 0$ where $\Delta = \partial_1^2 + \ldots + \partial_n^2$, $\partial_k = \partial/\partial x_k$, is the Laplace operator. Dirichlet's problem: given Ω and a continuous function f on the boundary Γ of Ω, find u harmonic in Ω and continuous in $\Omega \cup \Gamma$ such that $u = f$ on Γ. When $n = 1$, the harmonic functions are of the form $ax_1 + b$, and conversely, so that the reader may solve Dirichlet's problem by himself when Ω is an interval on the real axis. But if $n > 1$ we are in deep water. The physical examples that follow indicate that the problem is correctly posed in the sense that the solution is likely to exist and be uniquely determined by f and Ω, at least under some very light restrictions.

P. G. L. Dirichlet

Gravitation and electrostatics

A gravitational or electric potential in $\mathbb{R}^3$,

$$u(x) = \int |x - y|^{-1}\rho(y)dy$$

of a mass or charge with density ρ satisfies Poisson's equation

$$\Delta u(x) = -4\pi\rho(x),$$

as he proved in 1810. Hence, outside the masses or charges, u is harmonic. This can also be seen by differentiating under the integral sign and noting that $\Delta|x - y|^{-1} = 0$ for fixed $y \neq x$. Dirichlet's problem here becomes: find a potential in a region outside the masses (or charges in the electric case) when its value is known on the boundary of the region.

1 Expanded version of a lecture to a student audience at Lund University, Sweden.

2 The lack of precision at this point about the smoothness of functions and boundaries is intentional. Such vagueness was the rule in early nineteenth century mathematics (and in textbooks of not so long ago). Following the historical development, precision increases towards the end of the article.

Heat conduction

In his book *Theorie analytique de la chaleur* (1822), Fourier devised a mathematical model for the propagation of heat in a heat conducting body Ω. In this model, the temperature is represented by a function $u(t, x)$ of time t and position x in Ω which satisfies the heat equation $\partial_t u = \Delta u$. In a state of equilibrium, u is independent of t and hence harmonic. Dirichlet's problem becomes: compute the equilibrium temperature in Ω when its boundary Γ has a given time-independent temperature.

Elastic equilibrium

Let $u(x_1, x_2)$ be a smooth function defined in an open bounded part Ω of R^2 and think of the graph of u as a thin elastic membrane in space. Keep $u = f$ fixed over the boundary Γ of Ω. The potential energy of the membrane is supposed to be proportional to the stretching

$$\int_\Omega ((1 + |\operatorname{grad} u(x)|^2)^{1/2} - 1)\,dx_1 dx_2,$$

i.e., the area enlargement from u = constant. For small grad u, this is approximately half of the so-called Dirichlet integral

$$\int_\Omega |\operatorname{grad} u|^2 dx.$$

If v, another smooth function, vanishes on Γ, an integration by parts gives

$$\int_\Omega (\partial_1 u \partial_1 v + \partial_2 u \partial_2 v)\,dx = -\int_\Omega v \Delta u\,dx$$

so that

$$\int_\Omega |\operatorname{grad}(u+v)|^2 dx = \int_\Omega |\operatorname{grad} u|^2 dx + \int_\Omega |\operatorname{grad} v|^2 dx - 2\int_\Omega v \Delta u\,dx. \tag{1}$$

If u is harmonic, the last integral vanishes and we make the following observation, called Dirichlet's principle: of all functions w $(= u + v)$ on Ω, equal to f on the boundary Γ, the solution u of Dirichlet's problem has the least energy $\int_\Omega |\operatorname{grad} w|^2 dx$. By the laws of mechanics, this means that the corresponding membrane is in a state of equilibrium.

Poisson and Green

Before 1825, Poisson had found simple explicit solutions of Dirichlet's problem for a ball and a disk. For the ball $|x| < R$, $n = 3$,

$$u(x) = (4\pi R)^{-1} \int_{|y|=R} (R^2 - |x|^2)|x-y|^{-3} f(y)\,dS(y), \tag{2}$$

where $dS(y)$ is the element of area on the sphere $|y| = R$. For the disk $|x| < R$, $n = 2$,

$$u(x) = (2\pi R)^{-1} \int_{|y|=R} (R^2 - |x|^2)|x-y|^{-2} f(y)\,ds(y), \tag{3}$$

where $ds(y)$ is the element of arc length on the circle $|y| = R$.

I will not go into Poisson's now obsolete proofs. Instead, I shall sketch how Green found an analogue of (2) for general regions. His construction is to be found in his famous 1828 paper entitled *An essay on the application of mathematical analysis to the theory of electricity and magnetism*, where he also proves the well-known Green's formula.

Green studied electrical potentials in three dimensions,

$$V(x) = \int |x-y|^{-1} \rho(y)\,dy.$$

Integrating the identity with arbitrary u,

$$|x-y|^{-1}\Delta u(y) = \operatorname{div}_y(|x-y|^{-1} \operatorname{grad} u(y) - u(y) \operatorname{grad}_y |x-y|^{-1}),$$

with respect to y over a bounded region Ω minus a small ball around $x \in \Omega$ and letting the radius of the ball tend to zero, he showed that

$$4\pi u(x) = -\int_\Omega |x-y|^{-1}\Delta u(y)\,dy + \int_\Gamma u(y) \frac{d}{dN}|x-y|^{-1} d\Gamma(y) - \int_\Gamma |x-y|^{-1} \frac{du(y)}{dN} d\Gamma(y), \tag{4}$$

where Γ is the boundary of Ω, $d\Gamma(y)$ its element of area, and d/dN the interior normal derivative at y (when x is outside of Ω, the right side is zero). If u is harmonic, $\Delta u = 0$, this formula gives us u in Ω when we know u and du/dN on the boundary. Green observed that if one could find a function $V(x, y)$, defined for x, y in Ω and without singularities there, such that the functions $y \to V(x, y)$ are harmonic and the function for fixed x,

$$G(x, y) = |x-y|^{-1} - V(x, y),$$

(now called Green's function for Ω with pole at $x \in \Omega$) vanishes on Γ, then the way is open to a solution of Dirichlet's problem. For if we do the computation leading to (4) again, but now with $|x-y|^{-1}$ replaced by $G(x, y)$, the last integral of (4) vanishes, and if u is harmonic we get

$$u(x) = \int_\Gamma P(x, y) u(y)\,d\Gamma(y), \tag{5}$$

where

$$P(x, y) = (4\pi)^{-1}dG(x, y)/dN. \qquad (6)$$

Poisson's solutions (2) and (3) of Dirichlet's problem have this form and therefore $P(x, y)$ is called the *Poisson kernel* of Ω. The problem is now to show that the region Ω has a Green's function and that the formula

$$u(x) = \int P(x, y)f(y)d\Gamma(y) \qquad (7)$$

defines a harmonic function in Ω, equal to f on Γ. Green found the solution in a physical principle: when electrically charged conducting bodies are in electrostatic equilibrium, the sum of the potentials of all the charges is constant in each body. He thought of space outside Ω as an electric conductor with a negative charge $-\rho_x$ induced by a positive unit charge at the point x in Ω. The potential of the latter is $y \to |x - y|^{-1}$ and, if the potential of $-\rho_x$ is $y \to - V(x, y)$, their sum, the equilibrium potential $y \to G(x, y) = |x - y|^{-1} - V(x, y)$, is positive on Ω and harmonic there except at the point x and vanishes outside Ω. But these are just the desired properties. Outside $\Omega \cup \Gamma$, the potential $y \to G(x, y)$ is zero and hence harmonic so that the charge ρ_x is concentrated to Γ.

If $\rho(x, x')$ is the charge density of ρ_x on Γ, we have

$$V(x, y) = \int_\Gamma |y - z|^{-1}\rho(x, z)d\Gamma(z)$$

and can write Green's function as

$$y \to G(x,y) = |x - y|^{-1} - \int_\Gamma |y - z|^{-1}\rho(x, z)d\Gamma(z). \qquad (8)$$

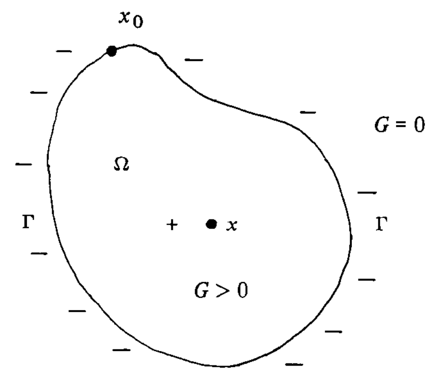
Since $G(y, z) = 0$ when y is in Ω and z on Γ, the first factor of the integrand equals

$$|y - z|^{-1} = \int_\Gamma |z - \zeta|^{-1}\rho(y, \zeta)d\Gamma(\zeta)$$

and an insertion into (8) shows Green's function to be symmetric, i.e., $G(y, x) = G(x, y)$. In particular, all functions $x \to G(x, y)$ are harmonic in Ω outside the point y. Hence, by (6), the functions $x \to P(x, y)$ are harmonic in Ω so that, by differentiation under the sign of integration, the right side of (7) is harmonic there for every continuous f.

It remains to show that the function u as defined by (7) tends to f at the boundary of Ω. To see this, go back to Green's function $x \to G(x, y)$. Using its symmetry, it follows that $G(x, y) = 0$ when $x \neq y$ and when x or y or both are on Γ. Hence, by (6), $x \to P(x, y)$ vanishes on Γ outside of y. Moreover, $P(x, y) \geqslant 0$ and, putting $u = 1$ in (4) we get

$$\int_\Gamma P(x, y)d\Gamma(y) = 1$$



Green's function $y \to G(x, y)$ is the equilibrium potential of a unit positive charge at $x \in \Omega$ and an induced negative charge $-\rho_x$ on the boundary Γ. As $x \to x_0 \in \Gamma$, ρ_x tends to the unit charge at x_0.

for all x in Ω. Hence, when $x \to x_0 \in \Gamma$, the charge on Γ with the density $y \to P(x, y)$ tends to a unit charge at x_0. This shows that $u(x) \to f(x_0)$ as $x \to x_0$.

The last piece of this wonderful line of beautiful arguments is the proof that the Poisson kernel $P(x, y)$ is nothing but the density $\rho(x, y)$ of the induced charge ρ_x. This means that (7) can be written as

$$u(x) = \int_\Gamma f(y)\rho(x, y)d\Gamma(y),$$

where ρ has an immediate physical significance. Green shows that the potential

$$V(y) = \int_\Gamma |y - x'|^{-1}\rho(x')d\Gamma(x')$$

of an arbitrary charge density $\rho(x')$ on Γ has the property that the sum of the two normal derivatives of V at $x' \in \Gamma$, both directed away from Γ, is $-4\pi\rho(x')$. Applied to (8) this shows that (note that $y \to G(x, y)$ is zero outside of Ω)

$$\rho(x, x') = (4\pi)^{-1}dG(x, x')/dN$$

where x is in Ω and x' is on Γ and d/dN is the interior normal derivative at x'. For a ball $B : |x| < R$, there is an explicit formula for Green's function

$$G(x, y) = |x - y|^{-1} - R|x|^{-1}|x^* - y|^{-1},$$

where $x, y \in B$ and $x^* = R^2x/|x|^2$ is the image of x under a reflection in the sphere $|x| = R$. We see that $y \to G(x, y)$ is harmonic, that $G(x, y) = 0$ when $x = x^* \neq y$, and, by a simple computation, that $G(x, y) = G(y, x)$. Insertion into (6) gives the Poisson kernel

$$P(x, y) = (4\pi R)^{-1}(R^2 - |x|^2)|x - y|^{-3}$$

in accordance with (2). When $n = 2$, we have to exchange the electrostatic for the logarithmic potential,

$$V(x) = \int \log |x - y|^{-1} \rho(y) dy$$

with the property that $\Delta V(x) = -2\pi\rho(x)$. Green's function for the disk $|x| < R$ is

$$G(x, y) = \log R^{-1} |x| |x^* - y| |x - y|^{-1}$$

and a computation gives the Poisson kernel

$$P(x, y) = (2\pi R)^{-1}(R^2 - |x|^2)|x - y|^{-2}$$

in accordance with (3).

Gauss

Gauss' contribution to potential theory is an 1839 paper in which he treats the subject from the beginning. Among other things he gives strict proofs of Poisson's equation and, without knowing Green's work, the basic properties of the derivatives of potentials of surface charges. This lucid and well-written work got many readers, among them the English physicists Thomson and Stokes. Both wrote about potential theory and through them Green's work became known in Germany where it was translated.

Gauss did not treat Dirichlet's problem but he invented an existence proof for the equilibrium potential of a charged surface. Let ρ be a charge on a surface Γ with density $\rho(y) \geqslant 0$ and let

$$V(x) = \int_\Gamma |x - y|^{-1} \rho(y) d\Gamma(y)$$

be its potential. Gauss introduced a quadratic form in ρ, namely

$$\begin{aligned} I(\rho) &= \int_\Gamma V(x)\rho(x) d\Gamma(x) \\ &= \int_\Gamma \int_\Gamma |x - y|^{-1} \rho(x)\rho(y) d\Gamma(x) d\Gamma(y). \end{aligned}$$

If σ is another charge, then

$$I(\rho + \sigma) = I(\rho) + 2 \int V(x)\sigma(x) d\Gamma(x) + I(\sigma).$$

Hence the function $\epsilon \to I(\rho + \epsilon\sigma)$ has the derivative

$$2 \int V(x)\sigma(x) d\Gamma(x) \tag{9}$$

at $\epsilon = 0$. If ρ is an equilibrium charge, i.e., if V is constant on Γ, this derivative vanishes for every charge σ whose total mass $\int \sigma(x) d\Gamma(x)$ is zero. Gauss observed that this characterizes the equilibrium charge: if, among all charges $\geqslant 0$ with given total mass, I takes its least value for a charge ρ, then its potential V is constant on Γ. The proof is simple. First Gauss remarks that if V attains its largest value on Γ at a point x', then $\rho(x') > 0$. (This follows from the maximum principle (see below) but Gauss gives a special proof.) If now $V(x') > V(x'')$ for some point x'' in Γ, we can get a negative derivative (9) by choosing a σ of total mass zero such that $\sigma(x) < 0$ close to x', $\sigma(x) > 0$ close to x'' and $\sigma(x) = 0$ otherwise. This contradiction shows that V is constant on Γ. Gauss also shows that the equilibrium charge is unique. With this he has given a mathematical proof that surfaces have equilibrium potentials. There is just one catch: we only have Gauss's word that there *is* a charge giving I its least value. A hundred years later, the proof was to be made rigorous, but just twenty years after it was published Gauss' proof was ripe for criticism. It could have been put forward by Weierstrass but he chose an easier target for his attack: Dirichlet's principle as it had been used by Riemann.

Riemann and Weierstrass

When $n = 2$, then $\Delta = 4\partial\bar{\partial}$ where $2\partial = \partial_1 - i\partial_2$, $2\bar{\partial} = \partial_1 + i\partial_2$ and therefore the harmonic functions are closely connected with the analytic ones, complex functions satisfying the Cauchy–Riemann differential equation,

$$\bar{\partial} f(x) = 0 \tag{10}$$

in open regions. If f is analytic, then $\Delta f = 4\partial\bar{\partial} f = \Delta \operatorname{Re} f + i \Delta \operatorname{Im} f = 0$ and hence $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are harmonic. Writing $z = x_1 + ix_2$ enables analytic functions to be thought of as maps $z \to f(x) = f(z)$ from the complex plane to itself. To motivate this and to understand the meaning of (10) we observe that the differential $df(z) = \partial_1 f(z) dx_1 + \partial_2 f(z) dx_2$ can be written as $\partial f(z) dz + \bar{\partial} f(z) d\bar{z}$. Hence, that f is analytic means in particular that the differential $dz \to \partial f(z) dz$ of the map $z \to f(z)$ is a linear similarity transformation when $\partial f(z) \neq 0$. At such points, the map $z \to f(z)$ is said to be *conformal*. The old term, similar in the least parts, is perhaps more expressive.

The simplest analytic function is $f(z) = z = x_1 + ix_2$, giving the identity map. Directly from the definition (10) follows easily that if $f(z)$ and $g(z)$ are analytic, so are $f(z) + g(z)$, $f(z)g(z)$, $f(z)/g(z)$, and $f(g(z))$ wherever they are defined. A direct verification shows that $\log z = \log|z| + i \arg z$ is analytic for $z \neq 0$ and hence $\log f(z)$ is analytic when $f(z) \neq 0$. The theory of analytic functions shows $f(z)$ to be analytic near $z_0 = x_{10} + ix_{20}$ if and only if $f(z)$ has a power series development around z_0,

$$f(z) = \Sigma a_n (z - z_0)^n,$$

convergent when $|z - z_0|$ is small enough. We shall not need this fact but we have to observe, finally, that if $f = u + iv$ with real u and v, then (10) can be written $\partial_1 u(x) = \partial_2 v(x)$, $\partial_2 u(x) = -\partial_1 v(x)$ so that $dv(x) = -\partial_2 u(x)dx_1 + \partial_1 u(x)dx_2$. This means that, apart from an additive constant, an analytic function is determined by its real part.

This short presentation of analytic functions combines ideas of Riemann and Weierstrass who, at about the same time and independently of each other, created order in what was earlier a rather chaotic theory. Riemann's famous 1851 thesis, *Foundations of a General Theory of the Functions of a Complex Variable,* deals with functions with certain desirable properties which, except for a finite number of points, are analytic in a given plane region or, more generally, on a Riemann surface, i.e., a manifold composed by overlapping such regions. Riemann, who had heard Dirichlet's lecture in Berlin, found such functions by constructing their real parts, using variants of Dirichlet's principle (the term is Riemann's) now taken as an existence proof: among all functions with given values on the boundary of a plane region Ω there *is* a function u with minimal Dirichlet integral. For every ϵ and function v vanishing on the boundary of Ω we then have

$$\int_\Omega |\text{grad}\,(u + \epsilon v)|^2 dx \geqslant \int_\Omega |\text{grad}\, u|^2 dx$$

and a comparison with (1) shows that, for all such v,

$$\int_\Omega v(x)\,\Delta u(x)dx = 0$$

which is possible only when u is harmonic. With such methods Riemann proved, among other things, the following result called the Riemann mapping theorem. Given an open bounded simply connected region Ω in $\mathbb{R}^2$, there is a conformal bijection $f(z)$ from Ω to the unit disk $|z| < 1$ such that $f(z_0) = 0$ at a point z_0 in Ω, given in advance. Now $f(z) = (z - z_0)g(z)$ where $g(z)$ is analytic and nonzero so that

$$-\log |f(z)| = \log |z - z_0|^{-1} - \log |g(z)|$$

is harmonic when $z \neq z_0$ and vanishes on the boundary of Ω where $|g(z)| = 1$. We recognize here Green's function for Ω with its pole at z_0 and it is easy to see that if we can construct Riemann's mapping function for a region we can also solve Dirichlet's problem for it and conversely.

In spite of his dazzling successes with Dirichlet's principle, Riemann did not escape criticism. Nobody doubted his results but the validity of Dirichlet's principle became an open question. This happened in connection with an increased interest around 1860 in the notions of irrational number, continuity, and differentiability and the realization that the distinction between a minimum and a greatest lower bound may be important. This was the beginning of the ϵ–δ period that is still with us. The great protagonist of the movement was Karl Weierstrass who was a professor in Berlin and survived Dirichlet and Riemann by twenty years. In 1872 he shocked conservative mathematicians by constructing a continuous but nowhere differentiable function. Some years earlier he had remarked that Dirichlet's principle needs a proof. Already the title of the article, On the so-called Dirichlet Principle, indicates doubt. After having reconstructed what the deceased Dirichlet had actually said in his lectures, Weierstrass establishes that nowhere in Dirichlet's assumptions is there a statement to the effect that there *is* a function for which Dirichlet's integral has a least value. And he makes the stern conclusion: all that can be said is that the integral in question has a greatest lower bound. Weierstrass finishes by proving that Dirichlet's principle is not valid for a modified Dirichlet integral in one dimension, on the surface very similar to the original.

Schwarz and Neumann

Two mathematicians in the generation after Riemann found existence proofs for Dirichlet's problem without using the doubtful Dirichlet principle. One of them was Hermann Amandus Schwarz. In his studies of conformal mappings he found an explicit formula for the Riemann mapping function from regions bounded by polygons and with this he solved Dirichlet's problems for such regions. Schwarz also made a rather laborious passage to the limit for the general case and invented a way of solving Dirichlet's problem for the union of two overlapping regions when it can be solved for each of them. He was the first to give a rigorous treatment of Dirichlet's problem for circular disks, earlier solved by Poisson. In his paper, Schwarz observes that Poisson's formula shows that the solution of Dirichlet's problem is infinitely differentiable in the open disk even when it is only continuous on the boundary and makes a careful study of how the solution behaves at the boundary. He emphasizes that the value of the solution at the center is the mean of the values at the boundary. From what he says explicitly we can put together the following important result.

Theorem. *A harmonic function u in an open region is infinitely differentiable there and has the mean value property,*

$$u(x_1, x_2) = (2\pi)^{-1} \int_0^{2\pi} u(x_1 + r\cos\theta, x_2 + r\sin\theta)d\theta$$

for every closed disk $|y - x| \leqslant r$ *in the region.*

With disks and means over disks replaced by spheres $|y - x| = r$ in n dimensions and means over them, this theorem holds also for harmonic functions in n variables. The mean value property, proved and used by Gauss for $n = 3$, shows that a harmonic function in any number of variables must be constant close to any point where it has a local maximum. This proves the important maximum principle: if a harmonic function assumes its least upper bound inside an open region where it is defined, it is constant there when the region is connected. This principle and a corresponding minimum principle (change u to $-u$) give a strikingly simple uniqueness proof for solutions of Dirichlet's problem for bounded regions: if a harmonic function vanishes on the boundary, it cannot be $\neq 0$ inside.

Carl Neumann, Schwarz's rival, found a completely different existence proof for Dirichlet's problem. He worked with two variables but the method extends to the general case. In Green's explicit formula for the solution,

$$u(x) = (2\pi)^{-1} \int_\Gamma f(y) dG(x,y)/dN\, ds,$$

where $y = y(s)$ ranges over Γ, s is the arc length and d/dN the interior normal derivative at y, Neumann replaced Green's function

$$G(x,y) = \log |x - y|^{-1} - V(x,y)$$

by its principal part, the logarithmic potential. Putting

$$Q(x,y) = (2\pi)^{-1} \frac{d}{dN} \log |x-y|^{-1}, \quad y = y(s),$$

he tried to find a solution $u(x)$ of Dirichlet's problem of the form

$$u(x) = \int_\Gamma Q(x,y) g(y) ds,$$

where g is a function to be determined. The function $x \to Q(x,y)$ is singular as x approaches y, but its limit $Q(x', y)$ as x tends to a point x' on Γ, not equal to y, has a smooth extension to all of Γ. Neumann proved that if x tends to any point x' on Γ, the integral tends to

$$g(x') + \int_\Gamma Q(x', y) g(y) ds.$$

In this way, Dirichlet's problem was reduced to finding a function g such that this expression equals the given function f. The equation for g can be written as an integral equation, namely

$$g(t) + \int_0^L K(t,s) g(s) ds = f(t), \tag{11}$$

where L is the length of Γ and, for simplicity, we have put $f(s) = f(y)$, $g(s) = g(y)$, and $K(t,s) = Q(x', y)$ when $y = y(s)$ and $x' = y(t)$. In the terminology of the turn of the century, this is an integral equation of the second kind for g with the kernel $K(t,s)$ (the first kind had no term $g(t)$; the use of the phrase Poisson kernel in connection with (2) and (3) is from the same time). We shall write (11) as an equation between functions,

$$g + Kg = f,$$

where

$$(Kg)(t) = \int_0^L K(t,s) g(s) ds.$$

Neumann solved this equation by successive approximations, $g_0 = f, g_1 = f - Kg_0, g_2 = f - Kg_1$ etc. which we write as

$$g_n = f - Kf + K^2 f + \ldots + (-1)^n K^n f,$$

a notation only implicit in Neumann's work. If the corresponding series, the so-called Neumann series,

$$f - Kf + K^2 f - K^3 f + \ldots,$$

formally analogous to the geometric series $(1-K)^{-1} = 1 - K + K^2 - K^3 + \ldots$, converges, it should be a solution of (11). Neumann proved that the series converges when the region Ω is strictly convex. Since Schwarz had solved Dirichlet's problem also for nonconvex regions, this was a victory with a sour note.

In this situation it became natural to study integral equations in general and to try to improve their theory beyond the Neumann series. Not only Dirichlet's problem could be reduced to such an equation. There was also a problem studied by Neumann and now named after him: find a function harmonic in a given region with a given normal derivative at the boundary. In 1900 Fredholm found a way of dealing with integral equations of the type (11). He approximated them by systems of linear equations with more and more unknowns. When the kernel $K(t,s)$ is continuous, as is the case for Dirichlet's problem when the boundary Γ has a continuously differentiable tangent, he could give explicit formulas for the solutions analogous to those obtained for systems of linear equations from the theory of determinants. They show that in several aspects the integral equation (11) behaves like a square system of linear equations, for instance so that if the homogeneous equation $g + Kg = 0$ has the unique solution $g = 0$, then (11) has a unique solution g for every right side f. For Dirichlet's problem, uniqueness is already taken care of and Fredholm could give an existence proof free of artificial assumptions. A few years after Fredholm, Hilbert, in a series of very influential papers, gave an abstract turn to the theory of integral equations. Nowadays the most important general theorems about the existence and uniqueness of solutions of integral equations do not depend on explicit formulas. They are part of functional analysis, the theory of linear spaces of infinite dimension and linear maps between them.

Rehabilitation of Dirichlet's principle and Green's existence proof – Potential theory

Weierstrass' criticism of Dirichlet's principle was not left unanswered. Poincaré (1887) and Hilbert (1898) turned it into a strict existence proof in two different ways. In both cases the boundary of the region is supposed to be suitably smooth. Poincaré's method only uses the maximum principle and Poisson's solution for balls. He noted that if u_1, $u_2, \ldots$ solve Dirichlet's problem with boundary values f_1, $f_2, \ldots$ converging uniformly to a continuous function f, then the solutions $u_1, u_2, \ldots$ converge uniformly to a solution u with boundary value f. One can therefore restrict oneself to very well behaved boundary values, e.g., a function f which is the restriction to Γ of a function F which is twice continuously differentiable in $\Omega \cup \Gamma$. If we replace this function F by a function F_B, equal to F outside of a closed ball B in Ω but harmonic in B and equal to F on the boundary of the ball, then the Dirichlet integral of F_B is at most that of F. Poincaré proved that iterations of the operation $F \to F_B$ for a suitably chosen infinite sequence of balls gives a sequence of functions converging to a solution of Dirichlet's problem. Choosing a colorful phrase, he named his procedure the method of sweeping out (méthode du balayage). The reason is that the function F can be thought of as a harmonic function plus a potential of a mass with the density $-\Delta F(x)/4\pi$. Replacing F by F_B means that the mass inside B that belongs to F is swept out to the boundary of the ball. The whole process, finally, is the sweeping out in steps of the mass belonging to F to the boundary of the region Ω. Hilbert, who treated Riemann's variants of Dirichlet's principle, made similar adjustments in a minimizing sequence for Dirichlet's integral resulting in a convergent sequence.

Using modern functional analysis and the Lebesgue integral, it is easy to see that Dirichlet's integral attains its greatest lower bound for a function with square integrable derivatives. Simple arguments prove it to be almost everywhere equal to a harmonic function which, under mild regularity assumptions on the boundary and the boundary function f, solves Dirichlet's problem. Applied in this manner, Dirichlet's principle works in all dimensions, for the equation of minimal surfaces, and for differential equations of higher order than 2, the elliptic ones, which appear for instance in the elasticity theory of solids.

The simplest known proof of the existence of a solution to Dirichlet's problem was invented by Perron in the 1920s and perfected by F. Riesz. It is a variant of the sweeping out method. Perron and Riesz consider continuous functions v on Ω which are $\geqslant f$ on the boundary Γ and superharmonic on Ω, i.e., the mean value over a sphere (in the plane a circle) in Ω is greater than or equal to the value in the center. It turns out that the pointwise greatest lower bound over this class V of functions,

F. Riesz

$$u(x) = \inf v(x) \quad \text{when} \quad v \in V,$$

is a solution of Dirichlet's problem for bounded regions with a smooth boundary.

Green's existence proof can also be made rigorous but then charges can no longer be defined by their densities. Instead one goes directly from physical intuition to a mathematical model that uses modern integration theory. Charges are considered as measures, i.e., functions $A \to \mu(A)$ from bounded sets in, e.g., $\mathbb{R}^3$ to the real line such that

$$\mu(A) = \mu(A_1) + \mu(A_2) + \ldots$$

where $A_1 \cup A_2 \cup \ldots$ is a partition of A, i.e., $A = A_1 \cup A_2 \cup \ldots$ and $A_j \cap A_k = \emptyset$ when $j \neq k$. The sets that are allowed are Borel sets which one obtains from the open and closed sets by taking countable unions and intersections. When $\mu(A) \geqslant 0$ for all A, we say that the measure is nonnegative, written $\mu \geqslant 0$. The *support* of a measure μ is the complement of the largest open set E where $\mu = 0$, i.e., $\mu(A) = 0$ for every $A \subset E$.

When h is a real continuous (or locally integrable) function and $\mu(A) = \int_A h(x)dx$ for open sets A, the measure is said to have the density h. Its support is then equal to the support of h. Not all measures have densities. To see this, take for instance the measure with the mass $a \neq 0$ at the point x_0 defined by $\mu(A) = a$ when x_0 is in A and $\mu(A) = 0$

otherwise. Its support is just the single point x_0. A measure can also be concentrated to lines and surfaces which means that its support is contained in these sets. A closed set is said to carry a measure when it contains the support of the measure.

The integral of a continuous function f with compact support with respect to a measure μ, the Riemann–Stieltjes integral

$$\int f(x)d\mu(x),$$

is defined as the limit of the Riemann sums

$$\Sigma f(x_k)\mu(A_k),$$

where $A_1 \cup A_2 \cup \ldots$ is a partition of $\mathbb{R}^3$ and the least upper bound of the diameters of $A_1, A_2, \ldots$ tends to zero. It is easy to see that the integral exists and has the usual properties. When μ has a continuous density h, then

$$\int f(x)d\mu(x) = \int f(x)h(x)dx$$

with an ordinary Riemann integral to the right. When $\mu = \mu_0$ consists of the mass 1 at the origin, then, clearly,

$$\int f(x)d\mu_0(x) = f(0).$$

We observe that if $\epsilon > 0$ and μ_ϵ has the continuous density $h(x/\epsilon)\epsilon^{-3}$ and $\int h(x)dx = 1$, then μ_ϵ tends to μ_0 as $\epsilon \to 0$ in the sense that

$$\int f(x)d\mu_\epsilon(x) = \int f(\epsilon x)h(x)dx \to f(0) = \int f(x)d\mu_0(x)$$

for every continuous function f with compact support.

When $\mu \geqslant 0$, and $f \geqslant 0$ is such that $f(x) = \lim f_n(x)$, where $f_1, f_2, \ldots$ is an increasing sequence of continuous functions with compact supports, the integral of f with respect to μ is defined by

$$\int f(x)d\mu(x) = \lim \int f_n(x)d\mu(x),$$

where the integrals on the right increase with n. The integral, finite or infinite, does not depend on the sequence $f_1, \ldots$ as long as it increases to f.

We can now define the potential of a measure $\mu \geqslant 0$, namely

$$U(x) = \int |x-y|^{-1}d\mu(x). \qquad (12)$$

It may be infinite but has certain regularity properties. It is continuous from below, i.e., if $U(x_0) > a$, then $U(x) > a$ when $|x - x_0|$ is small enough, and it is superharmonic which means that $U(x_0)$ is not less than the mean value of U on every sphere $S : |x - x_0| = r$ with its center at x_0. The first assertion is obvious, to prove the second one lets μ_S be a measure on S defined by the property that the integral

$$M_S(f) = \int f(y)d\mu_S(y)$$

is the mean of f on S, written as $\int_S f(y)dS(y)/4\pi r^2$ in classical notation, where $dS(y)$ is the area element. Its potential, the spherical potential

$$V_S(y) = \int |x-y|^{-1}d\mu_S(x)$$

is equal to $|x_0 - y|^{-1}$ outside of S and r^{-1} inside, a result of Newton. To compute $M_S(U)$, use (12) and change the order of the integrations which is allowed because all functions involved are $\geqslant 0$. The result is that

$$M_S(U) = \int V_S(y)d\mu(y) \geqslant U(x_0).$$

When U is the potential of a measure with a twice continuously differentiable density h of compact support, then

$$\begin{aligned}\int |\mathrm{grad}\, U(x)|^2 dx &= -\int U(x)\Delta U(x)dx \\ &= 4\pi \int U(x)h(x)dx \\ &= 4\pi \int\int |x-y|^{-1}h(x)h(y)dxdy \qquad (13)\end{aligned}$$

for here we are free to integrate by parts. Hence it is natural to consider the quadratic form I of Gauss, now extended to measures and denoted by

$$(\mu, \mu) = \int U(x)d\mu(x) = \int\int |x-y|^{-1}d\mu(x)d\mu(y), \qquad (14)$$

as a measure of the size or the energy of μ, at least when $\mu \geqslant 0$ so that the integral is well defined. In fact, the right side can be interpreted as twice the work done by collecting the mass (or charge) μ bit by bit from infinitely far away. The energy may be infinite (e.g., when the support of $\mu \neq 0$ is a point) but the energy of a spherical measure is clearly finite.

When $\mu, \nu \geqslant 0$ are two measures with finite energy, then their inner product

$$(\mu, \nu) = \int\int |x-y|^{-1}d\mu(x)d\nu(y),$$

a kind of mutual energy, is also finite and, as was shown by M. Riesz, the energy of their difference,

$$(\mu - \nu, \mu - \nu) = (\mu, \mu) - 2(\mu, \nu) + (\nu, \nu)$$

is $\geqslant 0$ and $= 0$ only when $\mu = \nu$.

We have seen that a sequence of measures with densities $\geqslant 0$ can converge to a measure without density. The great advantage of measures compared to measures with densities, simply expressed, is that limits of measures $\geqslant 0$ are also measures $\geqslant 0$. For instance, if $\mu_1, \mu_2, \ldots$ is a sequence of such measures and the numerical sequences

$$k \to \int V_S(x)d\mu_k(x)$$

converge for every spherical potential, then there is a measure $\mu \geqslant 0$ such that

$$k \to \infty \Rightarrow \int f(x)d\mu_k(x) \to \int f(x)d\mu(x)$$

for every continuous function f with compact support.

Measures and their potentials are the precision tools of modern potential theory. They provide rigorous mathematical models of the electrostatic equilibria whose existence Green took for granted and Gauss tried to prove. The two theorems below were first proved in the 1930s by de la Vallée Poussin and Frostman. The measures can be thought of as electric charges and the word mass can be read as charge.

Equilibrium with a given total mass (Equilibrium potential). *Let $F \subset \mathbb{R}^3$ be a compact set and suppose that F can carry measures $\omega \geqslant 0$ with positive and finite energy. Then the minimum energy of such measures of total mass 1, $\omega(F) = 1$, is attained by a unique $\omega = \nu$ whose potential $V(x) = \int |x-y|^{-1}d\nu(y)$ is constant $= C$ on F apart from a small exceptional set on the boundary of F where it is $< C$.*

Forced equilibrium (Induction). *Let $\mu \geqslant 0$ have finite energy, let $F \subset \mathbb{R}^3$ be closed and suppose that F can carry measures $\omega \geqslant 0$ with finite positive energy. Then the minimum energy of the difference $\mu - \omega$ is attained for a unique $\omega = \nu$ and if U, V are the potentials of μ, ν, then $V(x) \leqslant U(x)$ everywhere and $V(x) = U(x)$ on F apart from a small exceptional set on the boundary of F.*

The constant C of the first theorem has a physical significance: $1/C$ is the electrostatic capacity of F. Here we can only give a rough idea of the exceptional sets. They consist of points y in F which are irregular in the sense that $F \cap B$ is a very small part of a ball B with its center at y when its radius tends to zero. An isolated point in F is irregular but all points on, e.g., an ordinary hypersurface are regular. When F is the complement of a bounded open set Ω whose boundary Γ contains a point y irregular in F, there are complications in Dirichlet's problem for Ω. If f is continuous on Γ and $u(x)$ is the function on Ω given by Perron's method, it may happen for certain f that $u(x)$ does not have a limit as x tends to y. Such complications are inevitable, but the situation can be saved. In the 1920s Wiener showed that if f is extended to a continuous function F from Ω, Ω is approximated from the inside by regions Ω_n with regular boundaries Γ_n, and u_n is the solution of Dirichlet's problem for Ω_n, equal to F on Γ_m, then, as $n \to \infty$, u_n tends to a harmonic function u on Ω, uniquely determined by f and called the solution of the generalized Dirichlet problem. It is also the solution one gets directly by Perron's method.

When the set F of the forced equilibrium theorem contains a neighborhood of infinity and μ has compact support, the potentials $U(x)$ and $V(x)$ are equal for large x so that μ and ν have the same total mass. This is the case when F is the complement of an open bounded set Ω containing the support of μ. Outside of Ω, $V(x)$ is harmonic so that ν is supported by the boundary Γ of Ω. In Poincaré's words, ν is the result of sweeping out μ onto Γ. Following Green, we can also think of $-\nu$ as the charge on Γ, induced by μ. When $\mu = \mu_S$ is a spherical measure with center x and support in Ω, its potential is $y \to |x-y|^{-1}$ outside of the support, the induced charge $-\nu = -\nu_x$ only depends on x and the difference of potentials,

$$y \to G(x,y) = |x-y|^{-1} - \int |x'-y|d\nu_x(x')$$

is $\geqslant 0$ everywhere, positive and harmonic on Ω outside of x and zero off Ω except for a small exceptional set on Γ, empty when Γ is regular. This is Green's function in the form given by Green but now constructed for arbitrary bounded open sets. It will come as no surprise that

$$u(x) = \int f(y)d\nu_x(y)$$

is the solution of Wiener's generalization of Dirichlet's problem. Modern potential theory is the precision instrument that has given us the probably final analysis of Dirichlet's problem with continuous boundary values. Integration theory and the maximum principle also take care of the situation when the boundary values are measures. But the details of this and what the story is when the boundary values are distributions cannot be told here.

Generalized potential theory and Dirichlet spaces

Potential theory uses very few props, just measures μ, their potentials

$$\mu \to u(x) = \int H(x,y)d\mu(y)$$

and the energy

$$\int u(x)d\mu(x) = \int\int H(x,y)d\mu(x)d\mu(y)$$

where H stands for a kernel. The main results of the theory about equilibrium and induction use only these concepts. They are not limited to the classical potentials, Newtonian and logarithmic. Frostman proved in the 1930s that the theorems are true also for potentials with the kernel $H(x,y) = |x-y|^{\alpha-n}$, $0 < \alpha < 2$, in dimension $n > 1$ as suggested by M. Riesz. After that, other variants have been studied and there are axiomatized potential theories admitting other underlying spaces than just Euclidean ones.

The most imaginative and at the same time simplest axioms for a general potential theory have been invented by Arne Beurling. He combines Dirichlet's principle with what he calls (normal) contractions of the complex plane $\mathbb{C}$, i.e., maps $T\colon \mathbb{C} \to \mathbb{C}$ which preserve the origin, $T(0) = 0$, and do not increase distances, $|T(z_1) - T(z_2)| \leqslant |z_1 - z_2|$. Such contractions are, e.g., $z \to \bar{z}, z \to |z|$, and $z \to \operatorname{Re} z$. A fourth, and very important example is the projection T_1 on the interval $(0, 1)$ defined by $T_1(z) = \min(1, |z|)$. It follows from the definition of a derivative that $|(Tu(t))'| \leqslant |u'(t)|$ for complex differentiable functions of one variable. Hence, in all dimensions, the Dirichlet integral

$$I(u) = \int |\operatorname{grad} u(x)|^2 dx$$

has the property that $I(Tu) \leqslant I(u)$ for all T and u. This is Beurling's point of departure and allows him to introduce potentials without using kernels.

The axioms concern a linear space of generalized potentials, complex functions from a set X, and a Hilbert space D of such potentials, called a Dirichlet space. The norm square $||u||^2$ in D is interpreted as the energy of the potential u and it is required that $||Tu|| \leqslant ||u||$ for all u and contractions T. Already the case when X has a finite number of points $x, y, \ldots$ and D consists of all complex functions from E is very interesting. To have a norm square with the required contraction property, just take any positive definite sum

$$\Sigma\, b(x, y)|u(x) - u(y)|^2 + \Sigma\, c(x)|u(x)|^2$$

with coefficients $b, c \geqslant 0$. The corresponding inner product can be written

$$(u, v) = \Sigma\, a(x, y)u(x)\overline{u(y)}$$

with summation over $x, y \in X$ and unique coefficients $a(x, y) = \overline{a(y, x)}$. Since, by the contraction property, $||\bar{u}|| \leqslant ||u||$, we must have $||\bar{u}||^2 = ||u||^2$ which means that all $a(x, y) = a(y, x)$ are real. Following (13) we define Laplace's operator, now with a change of sign by the formula

$$\Delta u(x) = \Sigma\, a(x, y)u(y)$$

so that

$$(u, v) = \Sigma\, \Delta u(x)\bar{v}(x) = \Sigma\, u(x)\Delta v(x). \tag{15}$$

In this context Dirichlet's problem becomes: given a real-valued function f from a part Y of X, find a function u such that $u = f$ on Y and $u = 0$ outside. This is simply a square system of linear equations and we shall see that the contraction property gives uniqueness, existence and the maximum principle in one stroke.

Theorem. *The class V of potentials equal to f on Y has a unique element u of minimal norm which is also the unique solution of Dirichlet's problem. The maximum principle holds, i.e., all values of u lie between the extreme values of f.*

Proof. Since V is a convex part of D, there is a unique u in V of minimal norm, i.e., $||u||^2 \leqslant ||u + w||^2$ for all w vanishing on Y. Since the elements vanishing on Y form a complex linear subspace W of D, the inequality holds if and only if u is orthogonal to W. According to (15) with $v = w$, this property can be rewritten as $\Delta u(x) = 0$ outside of Y. Now, if in particular, $0 \leqslant f \leqslant 1$ then $T_1 V \subset V$ so that $||u|| \leqslant ||T_1 u||$. But the contraction property gives the opposite inequality and hence $T_1 u = u$, i.e., $0 \leqslant u \leqslant 1$. Letting $f = 0$ on Y except at one point y where $f(y) = 1$, we get a solution $x \to P_Y(x, y)$ of Dirichlet's problem with values between 0 and 1 which is the analogue of the Poisson kernel. Since $u(x) = \Sigma\, P_Y(x, y)f(y)$ is the solution in the general case, the proof is finished.

This theorem is taken from an article by Beurling and Deny (1959) which is completely elementary and has some illuminating discussions of the connections between the axioms of Dirichlet spaces and the fundamental results of potential theory.

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