Fachbereich Mathematik & Informatik
Freie Universität Berlin
Prof. Dr. Ralf Kornhuber, Xingjian Zhang-Schneider

Exercise 1 for the lecture

Numerics III

WS 2020

http://numerik.mi.fu-berlin.de/wiki/SS_2020/NumericsIII.php

Due: Friday, May 1st via Email

1. Exercise (5 TP)

Let U be a bounded, open subset of $\mathbb{R}^n$ and ∂U is C^1. Suppose $u \in C^1(\bar{U})$, then

$$\int_U u_{x_i} dx = \int_{\partial U} u\nu^i dS \quad (i = 1, \dots, n), \tag{1}$$

where we denote by ν^i the $i-th$ outward pointing unit normal vector. We write u_{x_i} for $\frac{\partial u}{\partial x_i}$. This result is often referred as the Gauss Theorem.

a) Prove the so called Divergence Theorem using the Gauss Theorem. The following equation

$$\int_U \text{div}\mathbf{u} dx = \int_{\partial U} \mathbf{u} \cdot \nu dS \tag{2}$$

holds for each vector field $\mathbf{u} \in C^1(\bar{U}, \mathbb{R}^n)$.

b) Prove the "Integration by parts formula" using the Gauss Theorem. Let $u, v \in C^1(\bar{U})$. Then

$$\int_U u_{x_i} v dx = -\int_U u v_{x_i} dx + \int_{\partial U} u v \nu^i dS \quad (i = 1, \dots, n). \tag{3}$$

c) Prove Green's formulas. Let $u, v \in C^2(\bar{U})$. Then

- $\int_U \Delta u dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS,$
- $\int_U \nabla v \cdot \nabla u dx = -\int_U u \Delta v dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u dS,$
- $\int_U u\Delta v - v\Delta u dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS.$

2. Exercise (3 TP)

Let $\Omega \in \mathbb{R}^3$ be a bounded domain with sufficient smooth boundary and

$$H_C = \{v \in C^1(\bar{\Omega}) \mid v|_{\partial\Omega} = 0\}. \tag{4}$$

Prove that the variational equality

$$u \in H_C : \quad \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_C \tag{5}$$

has at most one solution.

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjian@gmail.com. with a subject starting by [NumericsIII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.