Fachbereich Mathematik & Informatik
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Exercise 9 for the lecture

Numerics III

SS 2020

http://numerik.mi.fu-berlin.de/wiki/SS_2020/NumericsIII.php

Due: Friday, June 26th via Email

1. Exercise (3 TP)

a) Derive a weak formulation of the elliptic boundary value problem

$$-\Delta u + u = f \quad \text{in } \Omega, \qquad \frac{\partial}{\partial n} u = g \quad \text{on } \partial\Omega$$

and give sufficient conditions on f and g for existence and uniqueness of a weak solution.

b) Show that a weak solution $u \in C^2(\overline{\Omega})$ is a classical solution of this problem.

2. Exercise (3 TP)
Let $\Omega \subset \mathbb{R}^2$ be a domain with polygonal boundary and let $\mathcal{T}$ be a triangulation of Ω. Show that all $v \in S^{(m)}$ have weak derivatives $\partial_x v$ and $\partial_y v$.

3. Exercise (3 TP)
In the lecture, an approximation of the variational problem

$$u \in H_0^1(\Omega) : \quad a(u, v) = l(v) \quad \forall v \in H_0^1(\Omega)$$

was derived by using the finite element space S_h. The resulting variational problem

$$u_h \in S_h : \quad a(u_h, v) = l(v) \quad \forall v \in S_h$$

is rewritten as the linear system of equations

$$AU = b\ .$$

Show that A is positive definite if $a(\cdot,\cdot)$ is elliptic and that symmetry of $a(\cdot,\cdot)$ implies symmetry of A.

4. Exercise (3 TP)
Consider the boundary value problem

$$-u'' = f \quad \text{in } \Omega = (0,1), \qquad u(0) = u(1) = 0$$

with $f \in L^2(0,1)$.

a) Show existence and uniqueness of a weak solution $u \in H^1(0,1)$.

b) Let S_h denote the space of piecewise linear finite elements with respect to the partition $0 = x_1 < x_2 < \cdots < x_{N-1} < x_N = 1$ of $[0,1]$ and let u_h be the corresponding Ritz-Galerkin approximation of u. Show that

$$u_h(x_i) = u(x_i), \qquad i = 1, \ldots, N.$$

General remarks

You have to do the exercises in groups of up 3 people. Be prepared to demonstrate your solutions to theoretical problems at the given date in the tutorial. Solutions for programming problems have to be submitted via e-mail to xingjianz@gmail.com. with a subject starting by [NumericsIII] and denoting all members of the group. Please follow the additional advise for programming exercises on the homepage.