Inverse Problems Exercise 1
Sommersemester 2022
Return your written solutions either in person or by email
to vesa.kaarnioja@fu-berlin.de by Monday 2 May, 2022, 12:15

1. Let $A \in \mathbb{R}^{m\times n}$ have a singular value decomposition $A = U\Lambda V^{\mathrm{T}}$.

   (a) Show that the squares of the singular values of A are eigenvalues of the symmetric matrices $A^{\mathrm{T}}A \in \mathbb{R}^{n\times n}$ and $AA^{\mathrm{T}} \in \mathbb{R}^{m\times m}$. What are the corresponding eigenvectors? If $n > m$, what are the remaining $n - m$ eigenvalues of $A^{\mathrm{T}}A$? If $m > n$, what are the remaining $m - n$ eigenvalues of AA^{T}?

   (b) Verify the identity $\mathrm{Ran}(A)^{\perp} = \mathrm{Ker}(A^{\mathrm{T}})$ using the singular value decomposition.

2. Let $A \in \mathbb{R}^{m\times n}$ and recall how the Moore–Penrose pseudoinverse $A^{\dagger} \in \mathbb{R}^{n\times m}$ is defined. Prove (some of) the identities

$$A^{\dagger}AA^{\dagger} = A^{\dagger},$$
$$AA^{\dagger}A = A,$$
$$(A^{\dagger}A)^{\mathrm{T}} = A^{\dagger}A,$$
$$(AA^{\dagger})^{\mathrm{T}} = AA^{\dagger}.$$

   Using these, show that

$$A^{\dagger}A\colon \mathbb{R}^{n} \to \mathrm{Ker}(A)^{\perp},$$
$$AA^{\dagger}\colon \mathbb{R}^{m} \to \mathrm{Ran}(A)$$

   are orthogonal projections.

3. Consider the matrix equation $Ax = y$, where $A \in \mathbb{R}^{m\times n}$. The corresponding *least squares problem* is to find a *least squares solution* x_{LS} that minimizes the Euclidean norm of the residual, i.e.,

$$\|Ax_{\mathrm{LS}} - y\| = \min_{x\in\mathbb{R}^n} \|Ax - y\| = \min_{z\in\mathrm{Ran}(A)} \|z - y\|.$$

   (a) Show that $A^{\dagger}y$ is a least squares solution and satisfies the *normal equation*

$$A^{\mathrm{T}}Ax = A^{\mathrm{T}}y.$$

   Why is this solution special?

   (b) Show that $\mathrm{Ker}(A^{\mathrm{T}}A) = \mathrm{Ker}(A)$.

   (c) Use the above results to deduce that $x \in \mathbb{R}^n$ is a least squares solution if and only if it satisfies the normal equation.